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V. G. MAZ' YA, V. D. SAPOZHNIKOVA

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**Abstract**

**Full Text**

## **Reports of the Academy of Sciences of the USSR**

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**MATHEMATICS**

**V. G. MAZ' YA, V. D. SAPOZHNIKOVA**

### **SOLUTION OF THE DIRICHLET AND NEUMANN PROBLEMS FOR IRREGULAR DOMAINS BY METHODS OF POTENTIAL THEORY**

*(Presented by Academician V. I. Smirnov on 28 V 1964)*

In the paper <sup>(1)</sup> the results of Radon <sup>(2)</sup> concerning the double-layer potential were generalized. In the present note a generalization of Radon's theory is given in the part concerning the single-layer potential and the solvability of the Dirichlet and Neumann problems.

Below we shall use the notation and concepts introduced in <sup>(1)</sup>. Let us only recall that an open domain  $\Omega \subset E^3$ , bounded by a surface  $\Gamma$ , is considered. By  $\omega(P, \mathcal{E})$  we denote the solid angle under which the set  $\mathcal{E} \subset \Gamma$  is seen from the point  $P \in E^3$ , where, by definition,

$$\omega(P, P) = 2\pi - \omega(P, \Gamma \setminus P).$$

As in <sup>(1)</sup>, it is assumed that the absolute variation of the function of the set  $\omega(P, \mathcal{E})$  is bounded by a constant independent of  $P$  (condition (A)).

We note that condition (A) is necessary in the following sense: if the surface  $\Gamma$  is quadrable\* and there exist limiting values from within and from without of the double-layer potential with any continuous density, then this condition is satisfied, and moreover

$$\left| \sup_{P \in E^3 \setminus \Gamma} \text{var}_{E \subset \Gamma} \omega(P, \mathcal{E}) - \sup_{P \in \Gamma} \text{var}_{E \subset \Gamma} \omega(P, \mathcal{E}) \right| \leq 2\pi.$$

## 1°. The single-layer potential and its boundary flux.

The single-layer potential with charge  $\Phi(\mathcal{E})$ , where  $\Phi(\mathcal{E})$  is a completely additive function of Borel sets on  $\Gamma$ , is the integral

$$V(P) = \frac{1}{2\pi} \int_{\Gamma} \frac{1}{r_{PX}} \Phi(dX), \quad P \notin \Gamma.$$

It is obvious that  $V(P)$  is a harmonic function for  $P \notin \Gamma$ .

Let us define the interior boundary flux of a function  $u(P)$  harmonic in  $\Omega$ . Let  $\mathcal{K}(E^3)$  be the space of infinitely differentiable functions with compact supports in  $E^3$ . Suppose that for any function  $\varphi \in \mathcal{K}(E^3)$  and for any sequence of domains tending to  $\Omega$ , bounded by smooth surfaces  $\Gamma_m \subset \Omega$ , there exists the limit

$$l_u(\varphi) = \lim_{m \rightarrow \infty} \int_{\Gamma_m} \varphi(X) \frac{\partial u(X)}{\partial \nu} s(dX),$$

where  $\nu$  is the exterior normal to  $\Gamma_m$ . Suppose, in addition, that the functional  $l_u(\varphi)$ , defined on  $\mathcal{K}(E^3)$ , is bounded in  $C(\Gamma)$ . In this case the extension of  $l_u(\varphi)$  to  $C(\Gamma)$  can be represented by means of a completely additive set function  $\Sigma^{(i)}(\mathcal{E})$  in the form

$$\int_{\Gamma} \varphi(X) \Sigma^{(i)}(dX).$$

\* The quadrability of  $\Gamma$  makes it possible to define the solid angle  $\omega(P, \mathcal{E})$ , and together with it also the double-layer potential  $W(P)$ , for  $P \notin \Gamma$ .

We shall call this function of sets  $\Sigma^{(i)}(\mathcal{E})$  the **interior boundary flux** of the function  $u(P)$ . The exterior flux  $\Sigma^{(e)}(\mathcal{E})$  of the function  $u(P)$ , harmonic in  $C\bar{\Omega}$ , is defined analogously.

The assumptions made are in fact satisfied for the simple-layer potential; more precisely, the following holds:

**Theorem 1.** *A simple-layer potential with charge  $\Phi(\mathcal{E})$  has interior and exterior boundary fluxes, defined by the equalities*

$$\Sigma^{(i)}(\mathcal{E}) = -\Phi(\mathcal{E}) + \frac{1}{2\pi} \int_{\Gamma} \omega(X, \mathcal{E}) \Phi(dX),$$

$$\Sigma^{(e)}(\mathcal{E}) = \Phi(\mathcal{E}) + \frac{1}{2\pi} \int_{\Gamma} \omega(X, \mathcal{E}) \Phi(dX).$$

In studying the solvability of boundary-value problems, the following properties of the simple-layer potential are used.

**Lemma 1.** *Let  $V(P)$  be a simple-layer potential with absolutely continuous charge*

$$\Phi(\mathcal{E}) = \int_{\mathcal{E}} a(X) s(dX),$$

where  $a(X)$  is a bounded Borel-measurable function. Then at every point  $P \in \Gamma$  the limiting values of  $V(P)$  from outside and from inside  $\Omega$  exist and are equal.

**Lemma 2.** *If the interior and exterior limiting values of the potential  $V(P)$  on  $\Gamma$  exist and are equal, then  $V(P) \in \mathcal{L}_2^{(1)}(E^3)$ , where  $\mathcal{L}_2^{(1)}(E^3)$  is the closure of  $\mathcal{K}(E^3)$  in the metric of the Dirichlet integral, and the equality*

$$\int_{E^3} |\text{grad } V(X)|^2 dX = \frac{1}{\pi} \int_{\Gamma} V(S) \Phi(dS)$$

holds.

**2°. Solution of the Dirichlet and Neumann problems.** As shown in (1), the interior and exterior Dirichlet problems reduce, respectively, to the equations

$$W^{(i)} = f + Tf, \quad (D^{(i)})$$

$$W^{(e)} = -f + Tf, \quad (D^{(e)})$$

where the operator

$$(Tf)_S = \frac{1}{2\pi} \int_{\Gamma} f(X) \omega(S, dX)$$

acts in the space  $C(\Gamma)^*$ .

The interior (exterior) Neumann problem is posed as follows: find a function  $u(P)$ , harmonic in  $\Omega$  (in  $C\bar{\Omega}$ ), whose interior (exterior) boundary flux exists and coincides with a prescribed completely additive function of sets on  $\Gamma$ .

Thus the interior and exterior Neumann problems reduce to the functional equations

$$\Sigma^{(i)} = -\Phi + T^*\Phi, \quad (N^{(i)})$$

$$\Sigma^{(e)} = \Phi + T^*\Phi, \quad (N^{(e)})$$

where  $T^*$  is the operator adjoint to  $T$ .

\* It can be shown that the operator  $T$  is also bounded in the space of functions defined on  $\Gamma$

satisfying a Lipschitz condition of order  $\alpha \in (0, 1)$  in the sense of the metric  $E^3$ .

Introduce the notation:  $\Gamma_\varepsilon(S) = \{X; r_{XS} \leq \varepsilon\} \cap \Gamma$ . Let the surface  $\Gamma$  be such that

$$\limsup_{\varepsilon \rightarrow 0} \sup_{S \in \Gamma} \text{Var}_{\mathcal{E} \subset \Gamma_\varepsilon(S)} \omega(S, \mathcal{E}) < 2\pi$$

(condition (B)).

If condition (B) is satisfied, then the Fredholm radius of the operator  $T$  is greater than one. Therefore, if the surface  $\Gamma$  satisfies conditions (A) and (B), then for

equations  $(D^{(i)})$ ,  $(N^{(e)})$  (respectively,  $(D^{(e)})$ ,  $(N^{(i)})$ ), the Fredholm alternative is valid\*.

In the proof of uniqueness the following is used.

**Lemma 3.** *If  $\Gamma$  satisfies conditions (A) and (B), then the solutions of the homogeneous equations  $(N^{(i)})$  and  $(N^{(e)})$  generate simple-layer potentials continuous in  $E^3$ .*

Following the classical scheme, we obtain the final result.

**Theorem 2.** *If the surface  $\Gamma$  satisfies conditions (A) and (B), then: 1) the interior Dirichlet problem is solvable for any continuous function  $W^{(i)}(S)$ , and the exterior Neumann problem is solvable for any absolutely additive function  $\Sigma^{(e)}(\mathcal{E})$  of Borel sets; 2) the exterior Dirichlet problem is solvable for any continuous boundary function  $W^{(e)}(S)$ , and the interior Neumann problem is solvable for any absolutely additive function  $\Sigma^{(i)}(\mathcal{E})$  with zero total mass.*

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Leningrad State University  
named after A. A. Zhdanov

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## CITED LITERATURE

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<sup>2</sup> J. Radon, UMN, **1**, issue 3-4 (1946). <sup>3</sup> N. Dunford, J. T. Schwartz, *Linear Operators*, part 1, IL, 1962, p. 528.

\* We take this occasion to note that the expression for the Fredholm radius given in <sup>(1)</sup> is, generally speaking, incorrect.

*Note: Figure translations are in progress. See original paper for figures.*

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