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Abstract

Full Text

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MAJORANTS FOR THE DERIVATIVE OF A POLYNOMIAL

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1°. Let Π_n denote the set of all polynomials of degree $\leq n$. Let k be a fixed integer ≥ 1. For a given value of x, $-\infty < x < \infty$, put

$$M_k(x) = M_{k,n}(x) = \sup |P_n^{(k)}(x)|,$$

where $P_n^{(k)}(x)$ denotes, as usual, the k-th derivative, and the supremum is taken over all $P_n \in \Pi_n$ satisfying on the segment $[-1, 1]$ the inequality

$$|P_n(x)| \leq 1.$$

We shall call the function $M_k(x)$ the **majorant of V. A. Markov**, since for arbitrary k it was first studied by V. A. Markov (^{1}).

It is known that the computation of $M_k(x)$ is connected with great difficulties, which arise at those points x at which the extremal polynomials are the polynomials of E. I. Zolotarev (^{2}). Therefore there arises the important question of finding the asymptotic value of $M_k(x)$. This question was solved comparatively long ago by S. N. Bernstein.

S. N. Bernstein (^{3}) proved that at all interior points of the interval $[-1, 1]$ the asymptotic equality

$$M_{k,n}(x) \sim \left(\frac{n^2}{1-x^2}\right)^{k/2}.$$

holds.

A simple proof of the complete theorem of V. A. Markov, which concerns all points of the interval $[-1, 1]$, was given by S. N. Bernstein in (^{4}).

2°. Let a sequence of points be given

$$-1 \leq x_0 < x_1 < \cdots < x_n \leq 1. \qquad (\mathfrak{M}_n)$$

Put, for given k and x, $-\infty < x < \infty$,

$$N_k(x) = N_{k,n}(x, \mathfrak{M}_n) = \sup |P_n^{(k)}(x)|,$$

where the supremum is taken over all $P_n \in \Pi_n$ satisfying the inequalities

$$|P_n(x_j)| \leq 1, \qquad j = 0, 1, 2, \dots, n. \tag{1}$$

The present note is devoted mainly to the study of the connection between the functions $N_k(x)$ and $M_k(x)$.

3°. Concerning $N_k(x)$ we shall prove the theorem:

Theorem 1. For any point x and any sequence $(\mathfrak{M}_n)$ the equality

$$N_k(x) = \sum_{i=0}^{n} |l_i^{(k)}(x)|,$$

holds, where $\{l_i(x)\}_{i=0}^{n}$ are the fundamental Lagrange polynomials constructed for the sequence of numbers $(\mathfrak{M}_n)$. The extremal polynomial $Q_n(x)$, for which

$$N_k(x) = Q_n^{(k)}(x),$$

is uniquely determined from the conditions

$$Q_n(x_i) = \operatorname{sign} l_i^{(k)}(x), \qquad i = 0, 1, 2, \dots, n. \tag{2}$$

Proof. Let $R_n \in \Pi_n$ and let it satisfy inequality (1). By the Lagrange interpolation formula we have

$$R_n(x) = \sum_{i=0}^{n} R_n(x_i) l_i(x).$$

Consequently,

$$R_n^{(k)}(x) = \sum_{i=0}^{n} R_n(x_i) l_i^{(k)}(x).$$

Therefore, by virtue of (1), we obtain that

$$|R_n^{(k)}(x)| \leq \sum_{i=0}^{n} |l_i^{(k)}(x)|. \tag{3}$$

Let us now recall the definition of $N_k(x)$. In view of (3), one may conclude that

$$N_k(x) \leq \sum_{i=0}^{n} |l_i^{(k)}(x)|. \tag{4}$$

On the other hand, the polynomial $Q_n(x)$, which is uniquely determined by condition (2), satisfies the equalities

$$Q_n^{(k)}(x) = \sum_{i=0}^{n} Q_n(x_i) l_i^{(k)}(x) = \sum_{i=0}^{n} |l_i^{(k)}(x)|.$$

Thus, by the definition of $N_k(x)$, we have:

$$\sum_{i=0}^{n} |l_i^{(k)}(x)| \leq N_k(x). \tag{5}$$

It follows from inequalities (4) and (5) that

$$N_k(x) = \sum_{i=0}^{n} |l_i^{(k)}(x)| = Q_n^{(k)}(x).$$

Theorem 2. The function $N_k(x)$ has the following properties:

1) If the system of points $(\mathfrak{M}_n)$ is located symmetrically with respect to the origin, then $N_k(x)$ is an even function, $N_k(-x) = N_k(x)$.
2) $N_k(x)$ is continuous on the entire number axis.
3) The derivative $N_k'(x)$ exists at all points of the number axis, with the exception of the roots of the polynomials $\{l_i^{(k)}(x)\}_{i=0}^{n}$, where $N_k'(x)$ has discontinuities of the first kind. If $x_{j,i}^{(k)}$ is a root of the polynomial $l_j^{(k)}(x)$, then

$$|N_k'(x_{j,i}^{(k)} + 0) - N_k'(x_{j,i}^{(k)} - 0)| = 2|l_j^{(k+1)}(x_{j,i}^{(k)})|.$$

4°. It is obvious that at each point

$$M_k(x) \leq N_k(x). \tag{6}$$

It is natural to ask under what conditions the majorants coincide, i.e., $M_k(x) = N_k(x)$.

For the study of this question the following theorem is useful:

Theorem 3. Suppose that, for a given point x, there exist a polynomial $R \in \Pi_n$ and a sequence of points $\mathfrak{M}_{n,x}$ of the form $(\mathfrak{M}_n)$ such that:

1) $|R(x)| \leq 1, \quad -1 \leq x \leq 1$.

2) $R(x_i^{(n)}) = \pm(-1)^i, \quad i = 0, 1, 2, \dots, n$.

3) $\operatorname{sign} l_i^{(k)}(x) = -\operatorname{sign} l_{i+1}^{(k)}(x); \quad i = 0, 1, 2, \dots, n$.

Then at this point the equalities hold

$$M_k(x) = N_k(x) = |R^{(k)}(x)|. \tag{7}$$

Proof. According to Theorem 1,

$$N_k(x) = \sum_{i=0}^{n} \left| l_i^{(k)}(x) \right|.$$

We now use property 3) of the system of points $\mathfrak{M}_{n,x}$; then we obtain that

$$N_k(x) = \pm \sum_{i=0}^{n} (-1)^i l_i^{(k)}(x). \tag{8}$$

Let us now take into account that the polynomial R satisfies condition 2). Therefore equality (8) can be written in the form

$$N_k(x) = \pm \sum_{i=0}^{n} R(x_i) l_i^{(k)}(x).$$

Consequently,

$$N_k(x) = \left| R^{(k)}(x) \right|. \tag{9}$$

On the other hand, since $R \in \Pi_n$ and $|R(x)| \leq 1, \ -1 \leq x \leq 1$, it follows from the definition of the function $M_k(x)$ that

$$M_k(x) \geq \left| R^{(k)}(x) \right|. \tag{10}$$

Inequalities (10), (6), and (9) lead to equality (7).

5°. Let us now suppose that the system of points $(\mathfrak{M}_n)$ consists of the numbers

$$x_j = \cos \frac{n-j}{n} \pi, \qquad j = 0, 1, 2, \dots, n. \tag{$\mathfrak{M}_n^{(0)}$}$$

Denote the roots of the equations

$$\left((x+1)T_n'(x)\right)^{(k)} = 0, \qquad T_n(x) = \cos n \arccos x, \qquad \left((x-1)T_n'(x)\right)^{(k)} = 0$$

respectively by

$$\xi_1 < \xi_2 < \cdots < \xi_{n-k} \quad \text{and} \quad \eta_1 < \eta_2 < \cdots < \eta_{n-k}.$$

V. A. Markov (1) proved that the inequalities

$$\xi_1 < \eta_1 < \xi_2 < \eta_2 < \cdots < \xi_{n-k} < \eta_{n-k}$$

hold.

Put

$$E(M) = (-\infty, \xi_1] + [\eta_1, \xi_2] + \cdots + [\eta_{n-k-1}, \xi_{n-k}] + [\eta_{n-k}, \infty).$$

From V. A. Markov' s considerations (1) it follows that if, as the system of points $(\mathfrak{M}_n)$, one takes the system $(\mathfrak{M}_n^{(0)})$, and as $R(x)$ the polynomial $T_n(x)$, then at every point of the set $E(M)$ all the conditions of Theorem 3 are satisfied. Therefore, from Theorem 3 there follows

Theorem 4. At every point of the set $E(M)$ the equalities

$$M_k(x) = N_k(x) = \left|T_n^{(k)}(x)\right|$$

hold.

In connection with this theorem there arises the question of the relation between the majorants $M_k(x)$ and $N_k(x)$ on the set $CE(M) = Z \setminus E(M)$, where Z is the whole number axis. This question is answered by

Theorem 5. For every point $x \in Z \setminus E(M)$ there exists a system of points $\mathfrak{M}_{n-1}$, depending on x and k, such that

$$M_{k,n}(x) = N_{k,n-1}(x, \mathfrak{M}_{n-1}).$$

This theorem is obtained from the results of V. A. Markov and Theorem 3.

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CITED LITERATURE

[1] V. A. Markov, *On functions least deviating from zero on a given interval*, St. Petersburg, 1892.
[2] E. I. Zolotarev, Collected Works, **2**, Publishing House of the Academy of Sciences of the USSR, 1932.
[3] S. N. Bernstein, Collected Works, **1**, Publishing House of the Academy of Sciences of the USSR, 1952, p. 153.
[4] S. N. Bernstein, Collected Works, **2**, Publishing House of the Academy of Sciences of the USSR, 1954, p. 281.

Note: Figure translations are in progress. See original paper for figures.

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