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Abstract

Full Text

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ON UNDERDETERMINED AND OVERDETERMINED SYSTEMS OF DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

(Presented by Academician P. S. Aleksandrov on 28 I 1964)

Let us consider the general system of linear differential equations with constant coefficients

$$pu = w. \tag{1}$$

Here $p = p(D)$ is an arbitrary matrix of size $m \times l$, formed from polynomials in the operator $D = \left(i\frac{\partial}{\partial x_1}, \dots, i\frac{\partial}{\partial x_n}\right)$; $w = (w_1, \dots, w_m)$ and $u = (u_1, \dots, u_l)$ are, respectively, a known and an unknown vector-function.

Systems of this kind have been studied in detail only with square nonsingular matrices. Such systems do not cover all interesting cases of system (1) that it is desirable to study from a general point of view; for example, systems corresponding to the operator of exterior differentiation of differential forms are not systems of this special type.

For arbitrary systems (1), only the most general results have been obtained: a description of hypoelliptic systems ([5,4]), theorems on solvability and on the general form of solutions (when right-hand sides are present), and some others ([1,2]). In the present note we formulate a number of consequences of the results of ([2]), representing a more detailed investigation of general systems of the form (1). The theorems that we formulate pertain to the following range of questions. Let Ω be some domain in the space R^n. How is the space of solutions of (1) in a neighborhood of the boundary of Ω related to the space of solutions of (1) in the domain Ω itself?

To study the operators p appearing in (1), let us consider the following characteristics of such operators. Let A be the ring of all polynomials in the operator D, and let A^k be the direct product of k copies of this ring. To each matrix p we associate the A-homomorphism $p : A^l \to A^m$ (which we shall denote by the same letter), consisting in multiplication of vectors from A^l on the left by the matrix p. Conversely, to every A-homomorphism $A^l \to A^m$ there corresponds some matrix of size $m \times l$ with entries in A.

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For every finite A-module M, i.e. an A-module having a finite basis, in accordance with Hilbert' s theorem ([3]), one can write an exact sequence of A-homomorphisms, called a syzygy chain, of the form

$$0 \to A^{l_\delta} \xrightarrow{q_\delta} A^{l_{\delta-1}} \to \dots \to A^{l_1} \xrightarrow{q_1} A^{l} \xrightarrow{p} A^{m} \to M \to 0, \tag{2}$$

where $p, q_1, \dots, q_\delta$ are certain matrices with entries in A, and $M \simeq \operatorname{Coker} p$, with $\delta < n$. Applying to this sequence the functor $\operatorname{Hom}(\cdot, A)$, we obtain a sequence of adjoint homomorphisms

$$A^{m} \xrightarrow{p'} A^{l} \xrightarrow{q_1'} A^{l_1} \to \dots \to A^{l_{\delta-1}} \xrightarrow{q_\delta'} A^{l_\delta} \to 0, \tag{3}$$

since $\operatorname{Hom}(A^k, A) \simeq A^k$, and $\operatorname{Hom}(q, A) = q'$, where by q' we denote the matrix transposed to q. The sequence (3), generally speaking, is not exact, but only semiexact. In order to characterize the degree of its non-exactness, we construct syzygy chains for the A-modules

$$\operatorname{Ext}^i(\operatorname{Coker} p, A) = \operatorname{Ker} q_i' / \operatorname{Im} q_{i-1}', \quad i = 1, \dots, \delta, \qquad q_0 = p.$$

$$\dots \to A^{s_i} \xrightarrow{h_i'} A^{t_i} \to \operatorname{Ext}^i(\operatorname{Coker} p, A) \to 0. \tag{4}$$

The operators H_i occurring in these sequences are convenient for describing the properties of solutions of system (1).

The operator p and system (1) are called determined, respectively underdetermined, if the rank of the matrix p is equal to l, respectively less than l. In view of the exactness of (2), the operator p is determined if and only if $q_1 = 0$. By $N(p) \subset C^n$ we shall denote the algebraic variety of points $s \in C^n$ at which the rank of the matrix $p(s)$ is less than l.

It is clear that neither the operators q_i, nor, all the more, the operators H_i, are determined uniquely by the operator p. However, it can be shown that the properties of these operators which interest us are invariant with respect to their choice. In particular, all the operators H_i are always determined, and the varieties $N(H_i)$ do not depend on their choice. Our further considerations are equally valid for any operators H_i arising from the construction described above.

As an example, let us consider the exact sequence

$$0 \to A^{1} \xrightarrow{d_0} A^{n} \xrightarrow{d_1} A^{\binom{n}{2}} \xrightarrow{d_2} \dots \to A^{\binom{n}{n-2}} \xrightarrow{d_{n-2}} A^{n} \xrightarrow{d_{n-1}} A^{1} \to 0, \tag{5}$$

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where d_i is the operator of exterior differentiation of differential forms of order i, written in the form of a suitable matrix. Sequence (5) contains chains of syzygies of all the operators d_i. The sequence dual to (5) coincides with (5). Hence it follows that, for each operator d_i, all varieties $N(H_j)$, $j = 1, \dots, n-1-i$, are empty. These properties of the operators d_i are preserved if, in the corresponding matrices, the operators $\partial/\partial x_i$ are replaced by arbitrary polynomials in various groups of operators $\partial/\partial y_j$, $j = 1, \dots, \nu$. For example, the operators $\partial/\partial x_i$ may be replaced by the operators $\partial/\partial z_i$, $i = 1, \dots, n$.

Let us formulate several theorems describing the properties of underdetermined systems. Among the solutions of underdetermined systems there are always generalized functions with an arbitrarily small support and an arbitrarily high order of singularity. This fact makes it natural to study the properties of solutions of underdetermined systems by considering these solutions up to solutions with compact supports.

We shall say that an operator p is virtually hypoelliptic if every solution of the system

$$pu = 0 \tag{6}$$

in R^n can be made infinitely differentiable in a neighborhood of the origin by adding to it some solution of the same system with compact support. If the operator p is determined, i.e. if there are no nonzero solutions of (6) with compact supports, then this definition coincides with the known definition of hypoelliptic operators.

Theorem 1. *The operator p is virtually hypoelliptic if and only if the operator H_i is hypoelliptic.*

Theorem 2. *Let the operators $H_1, \dots, H_k$, $k \geqslant 1$, be hypoelliptic, and let $\Omega \subset R^n$ be a domain which is the union of a finite or countable number of convex domains Ω_i, no more than k of which intersect simultaneously.*

Then, for every closed set $K \subset R^n$ and every neighborhood ω of the set $K \cap \Omega$, every solution of (1) in Ω with right-hand side w, infinitely differentiable in Ω, is the sum of a solution of (1) in Ω, infinitely differentiable on $K \cap \Omega$, and a solution of (6) with support in ω.

Both theorems remain true if the condition of infinite differentiability of the first summand and of w is simultaneously replaced by the condition of analyticity, and the requirement that the operators $H_1, \dots, H_k$ be hypoelliptic by the requirement that they be elliptic.

Theorem 3. *If every solution of (6) in R^n can be made equal to zero in a neighborhood of the origin by adding a solution of the same*

systems with compact support, then $N(H_1) = \emptyset$, i.e. sequence (3) is exact at the term A^l. Conversely, suppose the sequence is exact at the terms $A^l, \dots, A^{l^{k-1}}$,

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$k \geqslant 1$; Ω, K, ω are sets from the condition of Theorem 2.

Then every solution of (6) in Ω is the sum of a solution of the same system, equal to zero in a neighborhood of K, and a solution with support in ω.

Corollary. Let Ω and Ω_i be domains from the condition of Theorem 2, and let the sequence (3) be exact at the terms $A^l, \dots, A^{l^{k-2}}$, $k \geqslant 2$. Then system (1) is solvable in Ω if it is solvable in each domain Ω_i.

We note that a criterion for the solvability of system (1) in a convex domain was obtained in ([1]) and ([2]).

We shall call an operator p overdetermined if its transpose is underdetermined. Substituting p' for p in the constructions of the sequences (2) and (4), instead of the operators q_i, H_i we obtain new operators, which we shall denote by q^i, H^i, $i = 1, \dots, \chi$. We shall use these operators to describe properties of overdetermined systems.

Let k be an arbitrary integer, $0 \leqslant k < n$, and let $x = (\xi, \eta)$, where $\xi = (x_1, \dots, x_k)$, $\eta = (x_{k+1}, \dots, x_n)$ (in the case $k = 0$, $\eta = x$). By Ω we denote an arbitrary convex domain in the subspace $\xi = 0$, and by Γ the intersection of the domain Ω and an arbitrary neighborhood of its boundary. Further, let $S \subset R^n$ be an arbitrary half-space, and let ω be an arbitrary convex domain in the subspace $\eta = 0$.

The following theorem generalizes the well-known fact that every solution of an equation with constant coefficients ($m = l = 1$) in a certain domain is estimated in terms of its values near the boundary of this domain.

Theorem 4. Let $N(H^i) = \emptyset$, $i = 1, \dots, k$. Then the mapping

$$\Phi(\Gamma \cap S \times \omega) \ni u \to u \in \Phi(\Omega \cap S \times \omega)/\Phi_0, \tag{7}$$

defined on solutions of (6) in the domain $\Omega \cap S \times \omega$, is continuous. Here $\Phi(G)$ is any of the spaces $\mathcal{D}'(G)$, $\mathcal{E}(G)$, $(S^\beta(G))'$, $\mathcal{E}^\beta(G)$ (see ([6,12])), and Φ_0 is the subspace of $\Phi(\Omega \cap S \times \omega)$ consisting of solutions of (6) with supports contained in $\Omega \cap S \times \omega$. Moreover, if the operator p is overdetermined, then the domain ω, and in the case $k = 0$ also the domain Ω, may be chosen nonconvex.

We shall say that the operator q satisfies the condition HE in the variables $x' = (x_1, \dots, x_t)$ if, from the fact that the function $q\varphi$ is infinitely differentiable in a neighborhood of the subspace $x' = 0$, and the function φ has compact support, it follows that the function φ is also infinitely differentiable in a neighborhood of this subspace. Obviously, every hypoelliptic operator satisfies the condition HE with respect to any group of variables.

Theorem 5. Let, for every $i = 1, \dots, k$, the operator H^i satisfy the condition HE in some $k - i + 1$ of the variables ξ.

Then every solution of (6) in $\Omega \cap S \times \omega$, infinitely differentiable in $\Gamma \cap S \times \omega$, can be made infinitely differentiable on any closed subset of $\Omega \cap S \times \omega$ by adding

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a solution of (6) with support in $\Omega \cap S \times \omega$. Moreover, if the operator p is overdetermined, then the domain ω, and in the case $k = 0$ also the domain Ω, may be chosen nonconvex.

From Theorem 5 it follows, in particular, that if p is an arbitrary overdetermined operator ($k = 0$), then every solution of (6) in $\Omega \cap S$ ($\eta = x$) is infinitely differentiable provided that it is infinitely differentiable in $\Gamma \cap S$. For operators with $l = m = 1$ this fact was established by M. S. Agranovich ([9]) for $S \supset \Omega$, and by F. John ([13]), B. Malgrange ([10]), and V. V. Grushin ([11]) for a half-space S in general position.

We shall say that an algebraic variety $N \subset C^n$ is hyperbolic in the direction s_1 if it lies in the domain $|s_1| \leqslant C|(s_2, \dots, s_n)| + B$,

$|\operatorname{Im} s_1| \leq C|\operatorname{Im}(s_2, \dots, s_n)| + C|s|^\gamma + B$ with some $C > 0$, $B > 0$, and $\gamma < 1$. We shall call a variety N elliptic in the direction s_1 if it lies in the domain $|\operatorname{Re} s_1| \leq C|\operatorname{Im} s| + C|s|^\gamma + B$ with some $C > 0$, $B > 0$, and $\gamma < 1$.

Theorem 6. *Let the operator p satisfy the following conditions. The variety $N(H^1)$ is elliptic with respect to x_n, and all operators $H^2, \dots, H^k$, $k \geq 1$, are definite. In addition, if $S \supset \Omega$, then a weaker condition of the following form is imposed. Decompose the module $\operatorname{Im} H^k$ into an intersection of irreducible finite A-modules $M_1, \dots, M_L$ (see (3)) and construct homomorphisms $g_\lambda : A^{r_\lambda} \to A^m$ in such a way that $\operatorname{Im} g_\lambda = M_\lambda$, $\lambda = 1, \dots, L$. The weakened condition is that each variety $N(g_\lambda)$ is not hyperbolic in at least one of the directions lying in the subspace of the variables ξ.*

Then every solution of (6) in $\Omega \cap S \times \omega$, equal to zero in $\Gamma \cap S \times \omega$, is equal to zero on any closed subset of $\Omega \cap S \times \omega$ with precision up to solutions of (6) with supports contained in $\Omega \cap S \times \omega$. If p is a definite operator, then the domain ω, and in the case $k = 0$ also the domain Ω, may be chosen nonconvex.

The following theorem demonstrates another property of overdetermined systems—the possibility of continuing, inside a domain, solutions of these systems that are prescribed beforehand only in a neighborhood of some part of the boundary of this domain.

Theorem 7. *Let $N(H^1) = \emptyset$, and let the operator q^i satisfy the condition of the preceding theorem.*

Then every solution of (6) in $\Gamma \cap S \times \omega$ can be continued into the domain $\Omega \cap S \times \omega$ as a solution of the same system. If the operator p is definite, then the domain ω may be chosen nonconvex. If, in addition, $k = 0$, then the domain Ω may also be taken nonconvex, but only under the condition that the domain $\Gamma \cap S$ is connected.

Thus, for example, if $p = d_i$ (then $k = n - 2 - i$), we obtain the following result from vector analysis: every closed form of order i, defined in a neighborhood of $\Gamma \cap S \times \omega$, can be continued to a closed form in $\Omega \cap S \times \omega$.

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If we put $p = d_0$, and in the matrix d_0 replace the operators $\partial/\partial x_i$ by the operators $\partial/\partial \bar{z}_i$, then system (6) becomes the Cauchy-Riemann system. The operator p is definite, and $N(H^j) = \emptyset$, $j = 1, \dots, n-1$. Consequently, Theorem 7 with $k = n-2$ and an arbitrary domain Ω such that the domain $\Gamma \cap \Omega$ is connected is applicable also to this system. Thus we obtain a strengthening of the Osgood-Brown theorem on the continuation of analytic functions. The special case of Theorem 7 with $k = l = 1$ and $S \supset \Omega$ was obtained by L. Ehrenpreis (7). In the case $k = 1$ and a half-space S in general position, this theorem was established independently by B. Malgrange (1) for convex domains and by V. V. Grushin jointly with the author for the case $l = 1$ in a somewhat stronger form (8).

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