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Abstract

Full Text

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PASSAGE OF A NONEQUILIBRIUM GAS THROUGH THE CRITICAL SECTION OF A NOZZLE

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The paper solves, in the first two approximations in small relaxation parameters, the hydrodynamic equations with allowance for relaxation processes for steady flows of a nonequilibrium gas in the neighborhood of the critical section of a nozzle and states: 1) close to equilibrium, 2) close to completely nonequilibrium (“frozen”).

The equations of flow of a nonequilibrium gas, when ordinary viscosity and thermal conductivity are neglected, have the familiar form

$$\partial\rho/\partial t + \operatorname{div}\rho\mathbf{v} = 0, \qquad \rho\, d\mathbf{v}/dt + \nabla p = 0, \qquad d\varepsilon/dt + p\, dV/dt = 0$$

(the notation is as in ([1])). Comparing the energy equation with the general thermodynamic expression

$$\frac{d\varepsilon}{dt} = -p\frac{dV}{dt} + \sum_n \varepsilon_n \frac{d\xi_n}{dt} + T\frac{ds}{dt},$$

where ξ_n are nonequilibrium parameters, we obtain the entropy balance equation

$$T\frac{ds}{dt} = -\sum_n \varepsilon_n \frac{d\xi_n}{dt}.$$

It may be assumed that the rates of change of the nonequilibrium parameters are determined only by the chemical potentials, i.e., the kinetic equations are valid

$$\frac{d\xi_n}{dt} = -\sum_m L_{nm}\varepsilon_m - \sum_{m,l} L_{nml}\varepsilon_m\varepsilon_l - \dots,$$

where $L_{nm}, L_{nml}, \dots$ are kinetic coefficients. The condition that entropy increase with time imposes certain restrictions on the forms written.

This system of equations is closed by expanding the dependent quantities (p, ε_n and T) in the independent parameters (V, ξ_n and s) to the required orders of smallness.

Let us consider a steady quasi-one-dimensional flow of a nonequilibrium gas through a nozzle in the neighborhood of the critical section. In view of the degeneracy of the hydrodynamic equations at the critical section, the expansions of the dependent quantities must be taken with terms of at least third order of smallness. Restricting ourselves to the simplest form of the expansion and taking, for simplicity, only one nonequilibrium parameter into account, we obtain

$$p' = p_V V' + p_\xi \xi' + \frac{1}{2} p_{VV} V'^2 + p_{V\xi} V' \xi' + \frac{1}{6} p_{VVV} V'^3 - T_V s'; \qquad (1)$$

$$\varepsilon_\xi = -p_\xi V' + \varepsilon_{\xi\xi} \xi' - \frac{1}{2} p_{V\xi} V'^2. \qquad (2)$$

Here the prime refers to deviations from the state in the critical section, where all thermodynamic parameters are evaluated; letter subscripts denote differentiation; the numbers in parentheses above the terms indicate their order of smallness.* The expansion for T', in view of the smallness of s', has the form $T' = 0$.

* Following (2), we take the order of smallness of p_ξ equal to 1/2. Then the order of ξ' is 3/2 and the order o

The form of expansions (1), (2) assumes that all derivatives are taken at fixed composition, i.e., are "frozen." The relations between the equilibrium and frozen values of the thermodynamic-parameter derivatives are obtained by substituting the equilibrium value ξ' from the condition $\varepsilon_\xi = 0$ into (1), and have the form

$$p_{\nu 0} - p_{\nu\infty} = p_\xi \varepsilon_\nu^{(1)} / \varepsilon_{\xi\xi} > 0, \qquad p_{\nu\nu 0} - p_{\nu\nu\infty} = 3 p_\xi p_{\nu\xi}^{(1)} / \varepsilon_{\xi\xi}, \qquad (3)$$

where the subscripts 0 and ∞ refer, respectively, to the equilibrium and frozen states. In the adopted cubic order of expansion, differences in the remaining derivatives may be neglected.

Introduce dimensionless variables $\eta = (z - z_*)/l$, $\sigma = S/S_*$, $\lambda = V/V_*$, $\mu = v/v_*$, $\nu = T_* s / p_* V_*$, $\pi = p/p_*$, where l is a characteristic dimension. Then the system of equations under study has the form

$$\frac{\mu}{\lambda} \sigma = 1, \qquad \chi \frac{d\mu}{d\eta} + \sigma \frac{d\pi}{d\eta} = 0, \qquad (4)$$

$$\pi-1=-\chi_{1\infty}(\lambda-1)+\frac{1}{2}\chi_{2\infty}(\lambda-1)^2-\frac{1}{6}\chi_3(\lambda-1)^3+\delta\left[1-\frac{1}{3}\frac{\Delta\chi_2}{\Delta\chi_1}(\lambda-1)\right]+\vartheta_1(\nu-1), \tag{5}$$

$$\theta\mu\frac{d\delta}{d\eta}=-\delta+\Delta\chi_1(\lambda-1)-\frac{1}{6}\Delta\chi_2(\lambda-1)^2, \tag{6}$$

$$\Delta\chi_1\frac{d\nu}{d\eta}=\theta\mu\left(\frac{d\delta}{d\eta}\right)^2, \tag{7}$$

where the relaxation time $\tau=(L\varepsilon_{\xi\xi})^{-1}$, or $\theta=v_*\tau/l$, has been introduced; $\chi=v_*^2/p_*V_*$, $\chi_1=-V_*p_V/p_*$, $\chi_2=V_*^2p_{VV}/p_*$, $\chi_3=-V_*^3p_{VVV}/p_*$, $\Delta\chi_1=\chi_{1\infty}-\chi_{10}$, $\Delta\chi_2=\chi_{2\infty}-\chi_{20}$, $\vartheta_1=-V_*T_V/T_*$, $\delta=p_\xi\xi'/p_*$. The asterisk subscript refers to the critical section.

A combination of equations (4)-(7) makes it possible to obtain an equation for the velocity gradient. The singular point of this equation determines the crisis of the nonequilibrium flow (see, for example, [(3)]).

The small relaxation parameter may be either θ or θ^{-1}. As $\theta\to 0$, the state of the gas is close to equilibrium; as $\theta^{-1}\to 0$, it is close to frozen. We shall seek solutions of system (4)-(7) in the form $f=\sum_{i=0}^{\infty}f^{(i)}\omega^i$, where ω is a small relaxation parameter, defined below for each situation. It is assumed that the orders of smallness of the thermodynamic parameters $\Delta\chi_1$ and $\Delta\chi_2$ are equal to the order of smallness of θ (or θ^{-1}), i.e., to 1.

For definiteness, consider a nozzle whose profile in the neighborhood of the critical section is approximated by an arc of a circle. Then

$$\sigma=1+2\eta^2+O(\eta^4),\qquad l=\sqrt{2R_*r_*}, \tag{8}$$

where R_* and r_* are, respectively, the radii of the circular arc and of the critical section.

1. State close to equilibrium. From (6) and (7), in the zeroth approximation ($\theta=0$) we obtain $\delta^{(0)}=\Delta\chi_1(\lambda^{(0)}-1)-\frac{1}{6}\Delta\chi_2(\lambda^{(0)}-1)^2$, $\nu^{(0)}=1$, and (5) has the natural form

$$\pi^{(0)}=1-\chi_{10}(\lambda^{(0)}-1)+\frac{1}{2}\chi_{20}(\lambda^{(0)}-1)^2-\frac{1}{6}\chi_3(\lambda^{(0)}-1)^3.$$

Using (8) and expanding the solutions of the system in η, we obtain, in particular, $\chi=\chi_{10}$, i.e., $v_*=c_0$: the equilibrium flow crisis in the critical section of the nozzle. The solutions have the form

$$\lambda^{(0)} = 1 + 2\sqrt{\alpha\eta} + \frac{2}{3}\alpha(4+\beta)\eta^2 + \ldots, \tag{9}$$

$$\mu^{(0)} = 1 + 2\sqrt{\alpha\eta} + 2\left[\frac{1}{3}\alpha(4+\beta) - 1\right]\eta^2 + \ldots, \tag{10}$$

$$\pi^{(0)} = 1 - 2\chi_{10}\sqrt{\alpha\eta} - 2\chi_{10}\left[\frac{1}{3}\alpha(4+\beta) - 1\right]\eta^2 + \ldots, \tag{11}$$

where $\alpha = \varkappa_{10}/\varkappa_{20}$, $\beta = \varkappa_3/\varkappa_{20}$ (as usual, we assume $p_{VV} > 0$ ([1])). These solutions assume that the values introduced above of all the functions in the critical section (V_*, v_*, p_*, s_*) are equilibrium ones.

In the next approximation (6) has the form

$$\varkappa_{10}\mu^{(0)}\left[1 - \frac{1}{3}\frac{\Delta\varkappa_2}{\Delta\varkappa_1}\left(\lambda^{(0)} - 1\right)\right]\frac{d\lambda^{(0)}}{d\eta} = -\delta^{(1)} + \Delta\varkappa_1\left[1 - \frac{1}{3}\frac{\Delta\varkappa_2}{\Delta\varkappa_1}\left(\lambda^{(0)} - 1\right)\right]\lambda^{(1)},$$

where the expansion is carried out in the natural parameter

$$\omega = \frac{\Delta\varkappa_1}{\varkappa_{10}}\theta = \frac{c_0\tau_0}{l} = \frac{c_0}{l}\frac{c_\infty^2 - c_0^2}{c_0^2}\tau = \frac{\zeta}{c_0\rho_* l} = \frac{1}{\mathrm{Re}} \ll 1, \tag{12}$$

where c_0 and c_∞ are, respectively, the equilibrium and frozen speeds of sound, $\zeta = \tau\rho_*(c_\infty^2 - c_0^2)$ is the coefficient of second viscosity ([1]), and Re is the corresponding Reynolds number. The order of smallness of ω (or Re^{-1}) is 2. Equation (7) has the form

$$\frac{dv^{(1)}}{d\eta} = \varkappa_{10}\mu^{(0)}\left[1 - \frac{1}{3}\frac{\Delta\varkappa_2}{\Delta\varkappa_1}\left(\lambda^{(0)} - 1\right)\right]^2\left(\frac{d\lambda^{(0)}}{d\eta}\right)^2.$$

The solutions (in the first terms) are as follows:

$$\lambda^{(1)} = \mu^{(1)} = \frac{2}{3}\alpha^{3/2}\left[7 + \beta - 3\vartheta_1 - 2\frac{\Delta\varkappa_2}{\Delta\varkappa_1}\right] - \ldots = \frac{4}{3}\frac{5 - 3\gamma}{(\gamma+1)^{3/2}} - \ldots, \tag{13}$$

$$v^{(1)} = 4\varkappa_{10}\alpha\eta + \ldots = \frac{4\gamma}{\gamma+1}\eta + \ldots, \tag{14}$$

$$\pi^{(1)} = -\varkappa_{10}\left(2\sqrt{\alpha} + \lambda^{(1)}\right) + \ldots = -\frac{2}{3}\frac{\gamma(13 - 3\gamma)}{(\gamma+1)^{3/2}} + \ldots, \tag{15}$$

where the expressions with $\gamma = c_p/c_v$ are given for an ideal gas. It is significant that for $\gamma < 5/3$, taking nonequilibrium into account increases the specific volume and the velocity and lowers the pressure in the critical section. The gradients of all quantities in this case must be smaller in absolute value than in the equilibrium state. For a monatomic gas ($\gamma = 5/3$) the coefficient of second viscosity is equal to 0, and the flow characteristics in the present treatment do not change.

Without giving the expression for the flow velocity at the singular point of the system (4)-(7), we note that it is obtained by formal differentiation of (5) at $\xi = \text{const}$, i.e., at the flow crisis its velocity is equal to the frozen speed of sound ([4]), and in the zeroth approximation the value ξ' is taken from the condition $\varepsilon_\xi = 0$. However, in contrast to ([4]), the crisis of the nonequilibrium flow occurs in the first approximation not in the critical section, but at the point with coordinate

$$\eta_{**} = \frac{z_{**} - z_*}{l} = \frac{1}{2}\frac{\Delta\varkappa_1}{\varkappa_{10}}\sqrt{\alpha} = \frac{c_0}{2V_*}\sqrt{\frac{1}{V_* p_{VV}}}\,\frac{c_\infty^2 - c_0^2}{c_0^2} > 0, \tag{16}$$

i.e., in the diffuser. It is interesting that the magnitude of the displacement in the first approximation is a purely thermodynamic parameter, not equal to zero even for $\theta = 0$. The paradoxical nature of the situation for an equilibrium gas is typical of the conclusions of relaxation hydrodynamics ([5,6]). The displacement has order of smallness 1.

It should be borne in mind that describing the flow of a nonequilibrium gas through a nozzle by the equations of ordinary hydrodynamics with allowance only for volume viscosity cannot be correct. This follows from the general fact that the very concept of the coefficient of second viscosity has meaning only for hydrodynamic equations in the linear approximation ([6]), whereas flow through a nozzle can be described only in higher approximations. In particular, result (16) cannot be obtained at all within the framework of ordinary hydrodynamics.

2. A state close to frozen. From (6) and (7), in the zeroth approximation ($\theta^{-1} = 0$) we obtain $\delta^{(0)} = \text{const} = 0$, $\nu^{(0)} = 1$, and (5) has the natural form

$$\pi^{(0)} = 1 - \varkappa_{1\infty}(\lambda^{(0)} - 1) + \frac{1}{2}\varkappa_{2\infty}(\lambda^{(0)} - 1)^2 - \frac{1}{6}\varkappa_3(\lambda^{(0)} - 1)^3.$$

Using (8) and expanding the solutions of the system in η, we obtain, in particular, $\varkappa = \varkappa_{1\infty}$, i.e. $v_* = c_\infty$—the frozen flow crisis in the critical section of the nozzle. The solutions have the form (9)-(11), with $\varkappa_{10}, \varkappa_{20}$ replaced by $\varkappa_{1\infty}, \varkappa_{2\infty}$. The choice of the integration constant $\delta^{(0)} = 0$ was made so that in the critical section not only the specific volume and velocity, but also the pressure, would be frozen.

In the next approximation, (6) has the form

$$\mu^{(0)}\frac{d\delta^{(1)}}{d\eta} = \varkappa_{1\infty}\left[1 - \frac{1}{6}\frac{\Delta\varkappa_2}{\Delta\varkappa_1}(\lambda^{(0)} - 1)\right](\lambda^{(0)} - 1),$$

where the expansion is carried out in the natural parameter

$$\omega = \frac{\Delta\varkappa_1}{\varkappa_{1\infty}}\theta^{-1} = \frac{l\tau_\infty^{-1}}{c_\infty} = \frac{l}{c_\infty}\frac{c_\infty^2 - c_0^2}{c_\infty^2}\tau^{-1}, \tag{17}$$

whose order of smallness is also equal to 2. It is not difficult to verify from (7) that the change in entropy is of order of smallness 5, i.e. a state close to frozen is more isentropic than a state close to equilibrium. Moreover, to obtain even the gradients of the first approximation it is in fact sufficient to use the quadratic approximation for p' and ε_ξ instead of the cubic approximation (1) and (2). The solutions have the form

$$\lambda^{(1)} = \mu^{(1)} = -\frac{1}{2}a\eta + \dots, \qquad \nu^{(1)} = \frac{4}{3}\varkappa_{1\infty}a\eta^3 + \dots, \quad \pi^{(1)} = \pi^{(1)}(0) + \frac{1}{2}\varkappa_{1\infty}a\eta - \dots, \tag{18}$$

where $\pi^{(1)}(0) = \text{const} = 0$ (see below) is the integration constant. Thus, a small deviation from the frozen state does not change the values of the characteristics in the critical section. The gradients of all quantities are smaller in absolute value than their frozen values.

From the general result $v_{**} = c_{\infty **}$, in view of $v_* = c_\infty$, it follows that the flow crisis occurs in the critical section. For this reason $\pi^{(1)}(0) = 0$ must hold.

In conclusion, we note that the results of the work are readily generalized to the case of an arbitrary number of nonequilibrium parameters. In particular, instead of the second-viscosity coefficient in the form $\zeta = \tau\rho_*(c_\infty^2 - c_0^2)$, there appears the generalized quantity

$$\zeta = \rho_* \sum_n \tau_n (c_{\infty n}^2 - c_0^2),$$

where τ_n is the relaxation time of the parameter ξ_n, and $c_{\infty n}$ is the equilibrium speed of sound at the same frozen parameter.

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Note: Figure translations are in progress. See original paper for figures.