ON THE QUESTION OF UNIQUENESS OF THE SOLUTION OF AN INTEGRAL EQUATION

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1964

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Abstract

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MATHEMATICS

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(Presented by Academician L. S. Pontryagin, 9 I 1964)

Consider the system of equations

$$x(t) = \int_a^t K[t, s, x(s)]\, ds + f(t), \tag{1}$$

$$K[t, s, x] = \{K_1(t, s, x_1, \dots, x_n), \dots, K_n[t, s, x_1, \dots, x_n]\},$$

$$f(t) = \{f_1(t), \dots, f_n(t)\},$$

$a \le s \le t \le b,\ x \in Q$ (Q is some open domain in R_n). If

$$\|K[t, s, x'] - K[t, s, x'']\| \le \omega[t, s, \|x' - x''\|], \tag{2}$$

$\omega[t, s, \xi]$ is nondecreasing in ξ, and $y \equiv 0$ is the unique solution of the equation

$$y(t) = \int_a^t \omega[t, s, y(s)]\, ds,$$

then (1) cannot have more than one solution. This scheme for studying the question of uniqueness is widely used, but it has essential shortcomings: first, the Banach norm usually employed in (2) does not allow one to take into account the individual properties of the components of K; second, (2) does not take into account the influence of f (we note that if, for example, $n = 1$, K does not depend on t and is nondecreasing in x, $f = \text{const}$, then there exists no more than a countable set of such f for which (1) may have more than one solution).

On the basis of ([1]), below we propose certain refinements of the scheme mentioned, which make it possible to take into account the influence of $f(t)$ and of the individual components of the vector function $K[t, s, x]$.

1. The inequality $z \geq y$ $(z > y)$ between n-dimensional vectors $z = \{z_i\}$, $y = \{y_i\}$ means that $z_i \geq y_i$ $(z_i > y_i)$, $i = 1, \dots, n$.

Let $z(t) = \{z_1(t), \dots, z_k(t)\}$ be a nonnegative vector function measurable on $[a, b)$, and let $\Omega_k(z, \tau)$ be the collection of such vector functions

$$\omega[t, s, y] = \{\omega_1[t, s, y_1, \dots, y_k], \dots, \omega_k[t, s, y_1, \dots, y_k]\}, \qquad t \in [a, b),$$

$s \in [a, \tau(t)]$, $\tau \in (a, b]$, $0 \leq y \leq z(s)$, that $\omega[t, s, 0] \equiv 0$, ω is nondecreasing in y, and for every nonnegative measurable $y(t) \leq z(t)$ the function

$$\int_a^\tau \omega[t, s, y(s)]\, ds$$

is measurable on $[a, b)$. If

$$z \geq \int_a^\tau \omega[t, s, z(s)]\, ds$$

and $\omega \in \Omega_k(z, \tau)$, then, by the Birkhoff-Tarski theorem (^{2}), on $[a, b)$ the equation

$$y(t) = \int_a^{\tau(t)} \omega[t, s, y(s)]\, ds$$

has a measurable solution $u_z \leq z$ such that $u \leq u_z$ for $t \in [a, b)$ for every measurable nonnegative solution $u \leq z$.

Let $1 = i_0 < i_1 < \cdots < i_k = n$; for the vector $x = \{x_1, \dots, x_n\}$ introduce the generalized norm (see (2)) by setting $\|x\|_k = \{\xi_0, \dots, \xi_{k-1}\}$, where ξ_j is the ordinary norm of the vector $\{x_{i_j}, x_{i_j+1}, \dots, x_{i_{j+1}-1}\}$.

Denote by $v_m(t, \psi)$ and $w_m(t, \psi)$, respectively, the lower and upper solutions of the system

$$y(t) = \int_a^t M_m^r[t, s, y(s)]\, ds + \psi(t)$$

of order $r \leq n$. The vector-function $M_m^r[t, s, y]$ is defined for $a \leq s \leq t < b$, $\|y\| < \infty$, is nondecreasing in y, and satisfies the Carathéodory condition (3).

2. Theorem 1. *Let*

$$M_1^n[t, s, x] \leq K[t, s, x] \leq M_2^n[t, s, x], \qquad a \leq s \leq t < b, \quad x \in Q, \tag{3}$$

and let there exist such z *and* $\omega \in \Omega_k(z, t)$ *that* $z \geq \|w_2(t, f) - v_1(t, f)\|_k$,

$$\|K[t,s,x'] - K[t,s,x'']\|_k \leq \omega[t,s,\|x'-x''\|_k]_t \tag{4}$$

for $a \leq s \leq t < b,\ v_1(s,f) \leq x', x'' \leq w_2(s,f)$.

If $z \geq \int_a^t \omega[t,s,z(s)]\,ds$ *on* $[a,b)$ *and* $u_z \equiv 0$ *for* $t \in [a, a+\varepsilon)$, *then system* (1) *has on* $[a, a+\varepsilon)$ *at most one continuous solution.*

Remark 1. Condition (3) serves to construct an a priori estimate of solutions. Such an estimate can be obtained in various ways, for example by replacing (3) with the condition

$$\|K[t,s,x]\|_r \leq M^r[t,s\|x\|_r].$$

Remark 2. The condition $u_z \equiv 0$ may be replaced by the condition: $u_z \geq 0$ and there exist such z_1 and $\omega_1 \in \Omega_k(z_1,t)$ that $z_1 \geq u_z$,

$$\|K[t,s,x'] - K[t,s,x'']\|_k \leq \omega_1[t,s,\|x'-x''\|_k]_t$$

for

$$a \leq s \leq t < b,$$

$$\|x'-x''\|_k \leq z_1, \qquad v_1(s,f) \leq x', x'' \leq w_2(s,f);$$

$$z_1 \geq \int_a^t \omega_1[t,s,z_1(s)]\,ds \quad \text{in } [a,b) \quad \text{and} \quad u_{z_1} \equiv 0 \quad \text{for } t \in [a, a+\varepsilon)$$

(cf. (4-6)).

3. Theorem 1 can be strengthened by replacing the generalized norm by a system of differentiable functionals $\varphi_i(x)$, if, for example, $K[t,s,x]$ does not depend on t, and f is differentiable. The generalization to the case of the functionals considered in (7) is carried out automatically.

Theorem 2. *Let* $\Gamma_i(x)$ *be the gradient of the functional* $\varphi_i(x)$ *at the point* x, $\varphi_i(0) = 0$, *and* $\varphi_i(x) > 0$ *for* $x \neq 0$ $(i = 1, \dots, k)$. *Suppose, further,*

$$M_1^n[s,x] \leq K[s,x] \leq M_2^n[s,x], \qquad s \in [a,b), \quad x \in Q, \tag{5}$$

and there exist such z *and* $\Omega_n(z,t)$ *that*

$$z \geq \|w_2(t,f) - v_1(t,f)\|_k,$$

$$(\Gamma_i(x'-x''),\ K[s,x']-K[s,x''])\leq\omega_i[s,\varphi_1(x'-x''),\dots,\varphi_k(x'-x'')]$$

$$(i=1,\dots,k)\quad\text{in the domain } s\in[a,b),\quad v_1(s,f)\leq x',x''\leq w_2(s,f).$$

If $z\geq\int_a^t\omega[s,z(s)]\,ds$ *in* $[a,b)$ *and* $u_z\equiv 0$ *for* $t\in[a,a+\varepsilon)$*, then system* (1) *has on* $[a,a+\varepsilon)$ *at most one continuous solution.*

The remarks to the preceding theorem remain valid here as well.

From Theorems 1 and 2 there follow new uniqueness criteria. We limit ourselves to the remark that Theorems 1 and 2 make it possible to consider the uniqueness criteria known to us (see, for example, ($^{4-8}$)) from a single point of view.

4. If it is required to establish the uniqueness of a known solution y, then in (4) and (5) one should replace x'' by y; this sometimes makes it possible to substantially lower the restrictions on the order of growth of the majorant ω caused by the requirement $u_z\equiv 0$. We confine ourselves to an example.

Theorem 3. *Let* $K[t,s,0]=f=0$*, and for* $a\leq s\leq t<a+\varepsilon$, $\|x\|<\delta$*, suppose that we have:*

$$\|K[t,s,x]\|\leq M^1[t,s,\|x\|],$$

$$\sum_{i=1}^{i_1-1}|K_i[t,s,x]|\leq p_1(s,x)\sum_{i=1}^{i_1-1}|x_i|+p_2(s,x)\,\theta_1\left(\sum_{i=i_1}^{n}|x_i|\right),$$

$$\sum_{i=i_1}^{n}|K_i[t,s,x]|\leq p_3(s,x)\,\theta_2\left(\sum_{i=1}^{i_1-1}|x_i|,\ \sum_{i=i_1}^{n}|x_i|\right),$$

where

$$0\leq p_i(s)\leq P_i(s)\quad\text{for } \|x\|\leq\omega(s,0),\qquad\int_a^{\varepsilon}P_i(s)\,ds<\infty\quad(i=1,2,3),$$

$$\theta_1(\xi),\ \theta_2(\xi,\eta)\ \text{do not decrease in } \xi,\eta,\qquad\theta_1(0)=\theta_2(0,0)=0.$$

If

$$\lim_{\alpha\to 0}\int_\alpha^\delta\frac{d\xi}{\theta_2(\theta_1(\xi),\xi)}=\infty,$$

then system (1) has on $[a,a+\varepsilon)$ *at most one solution.*

According to Theorem 3 the system

$$x_1 = \int_0^t \{\sin x_1(s) + (\ln|x_2(s)|)^{-1}\}\, ds,$$

$$x_2 = \int_0^t \{\exp(-|x_1(s)|^{-1}) + \ln|x_2(s)|(\exp x_2(s))\}\, ds$$

has a unique solution, although here $\|K\| = O(|\ln\|x\||^{-1})$, i.e. the order of growth of $\|K\|$ substantially exceeds the order of growth of Osgood' s function.

5. For the system

$$x(t) = \int_a^b K[t, s, x(s)]\, ds + f(t), \tag{6}$$

where the vector-function $K[t, s, x]$ is defined for $t \in [a, b]$, $x \in Q$, the following generalization of Santoro' s theorem (^{9}) is valid.

Theorem 4. *Let there exist such z and $\omega \in \Omega_k(z, b)$ that*

$$\|K[t, s, x'] - K[t, s, x'']\|_k \leq \omega[t, s, \|x' - x''\|_k],$$

$$t, s \in [a, b], \qquad \|x' - x''\|_k \leq z.$$

If

$$z \geq \int_a^b \omega[t, s, z(s)]\, ds, \qquad u_z \equiv 0 \text{ on } [a, b],$$

then system (6) cannot have two distinct solutions x, y, for which $\|x - y\|_k \leq z$.

As an example of an application of this theorem, we give an estimate of an interval $(k, n-k)$ of nonoscillation (^{10,11}) for the equation

$$\mathcal{L}[y] \equiv y^{(n)} + \sum_{i=0}^{r} p_i(t) y^{(k)} = 0, \qquad r \leq n-1.$$

Let $G(t, s, a, b)$ be the Green function of the problem $y^{(n)}(t) = 0$, $y^{(i)}(a) = y^{(j)}(b) = 0$, $i = 0, \ldots, k-1$, $j = 0, \ldots, n-k-1$; let $H_m(t, s)$ be functions continuous for $t, s \in [0, h]$ satisfying the inequalities

$$H_m(t, s) \geq \left|\frac{\partial^m}{\partial t^m} G(t, s, a, b)\right|, \qquad 0 \leq a \leq \min[t, s], \quad \max[t, s] \leq b \leq h;$$

let $z(t) = \{z_0(t), \dots, z_r(t)\}$ be such a vector-function that $z(t) > 0$ for $t \in (0, h)$, $\sum_{j=0}^{k-i}(z_i^{(j)}(0))^2 > 0$ for $i < k$, $z_i(0) > 0$ for $i \geq k$, $\sum_{j=0}^{n-k-i}(z_i^{(j)}(0))^2 > 0$ for $i < n-k$, and $z_i(h) > 0$ for $i \geq n-k$.

Theorem 5. If the moduli of the characteristic numbers of the $r \times r$ matrix with elements

$$a_{ij} \geq \max_t \int_0^h H_i(t,s)\, \frac{z_j(s)}{z_i(s)}\, |\rho_j(s)|\, ds \qquad (i, j = 0, \dots, r) \tag{7}$$

are less than unity, then $[0, h]$ is an interval of $(k, n-k)$-nonoscillation for the equation $\mathcal{L}[y] = 0$.

Remark. The interval of $(n-1, 1)$-nonoscillation is the interval of applicability of Chaplygin' s theorem for the equation $\mathcal{L}[y] = 0$ (10–12). For $k = n-1$, on the basis of Corollary 2 of Theorem 4 of paper (13), in (7) one may replace $|p_0(s)|$ by $\max[0, p_0(s)]$. Therefore, for $r = 0,\ z = t^{n-1}$, Theorem 5 yields a refinement of a theorem of A. Yu. Levin (14), and for $n = 2,\ r = 1,\ z = \{1, 1\}$, a refinement of the Vallée-Poussin theorem (15).

In conclusion, let us note that the assertions presented extend to equations in Banach spaces and to equations with retarded argument, as was reported at the Izhevsk seminar by A. I. Logunov and N. V. Shklyaeva; on the basis of Theorems 4 and 5, V. V. Ostroumov obtained a number of new propositions on the distribution of zeros of solutions of linear differential equations and on differential inequalities for the many-point Vallée-Poussin problem (14,16).

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Received
10 XII 1963

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Note: Figure translations are in progress. See original paper for figures.

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