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Abstract

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V. Yu. Sandberg

METRIZATION OF LIPSCHITZ SPACES

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In the note [1] the notion of a Lipschitz space was introduced; it occupies an intermediate position between the notions of a uniform space and a metric space. This means that every metric space is at the same time a Lipschitz space, and every Lipschitz space is automatically endowed with a uniform structure. Every uniform structure is generated by at least one Lipschitz structure. However, since there exist nonmetrizable uniform structures, there also exist nonmetrizable Lipschitz structures. In the present note necessary and sufficient conditions are indicated for the metrizability (in the usual sense and in the generalized metric of Antonovskii–Boltyanskii–Sarymsakov) of a Lipschitz structure. We note a change in terminology: a set $T \subset K(E)$ containing, together with each τ, all $\tau' < \tau$, and called in [1] a domain, is here called a **filled set**; the smallest filled set containing $B \subset K(E)$ is called the **filling** of the set B.

1. Definition 1. a) A Lipschitz structure $\mathfrak{F}$ defined on a set E is called **pseudometrizable** (**metrizable**) if on E there exists a pseudometric (metric) generating this structure. b) Suppose two pseudometrics ρ and ρ' are given on E, generating the structures $\mathfrak{F}$ and $\mathfrak{F}'$, respectively. The pseudometric ρ is said to be stronger than the pseudometric ρ' if the structure $\mathfrak{F}$ is stronger than the structure $\mathfrak{F}'$; if $\mathfrak{F} = \mathfrak{F}'$, then the pseudometrics ρ and ρ' are called equivalent.

It is clear that if a Lipschitz structure is pseudometrizable and separated, then it is metrizable.

Definition 2. a) A set $T \subset K(E)$ will be called **convex** if from $2\tau_1, 2\tau_2 \in T$ it follows that $\tau_1 + \tau_2 \in T$. b) We shall call T **monotone** if from $\tau_1 + \tau_2 \in T$ it follows that either $2\tau_1 \in T$, or $2\tau_2 \in T$.

It is easy to prove that the filling of a monotone set is a monotone set.

Let $T \subset K(E)$. Denote by ${}^1/_2 T$ the set of those classes $\tau \in K(E)$ for which $2\tau \in T$. The set $\frac{1}{2^n}T$ is defined by induction, setting

$$\frac{1}{2^{n+1}}T = \frac{1}{2}\left(\frac{1}{2^n}T\right).$$

It is easily proved:

Proposition 1. a) If T is convex, then ${}^1/_2T$ is also convex. b) If T is monotone, then ${}^1/_2T$ is also monotone. c) If T is a filled set, then ${}^1/_2T$ is also a filled set. d) If T is a filled set, then ${}^1/_2T \subset T$.

Definition 3. a) A structure $\mathfrak{F}$ is called **convex** if $\mathfrak{F}$ has a base consisting of convex filled sets. b) A structure $\mathfrak{F}$ is called **monotone** if $\mathfrak{F}$ has a base consisting of monotone sets.

Obviously, a metrizable structure is convex and monotone. However, one can give examples of convex but not monotone, and monotone but not convex, Lipschitz structures.

Theorem 1. For the metrizability of a separable Lipschitz structure $\mathfrak{F}$, defined on E, it is necessary and sufficient that there exist a convex, monotone, and full set $T \in \mathfrak{F}$ such that the sequence $\left\{\frac{1}{2^n}T\right\}$ forms a base of the filter $\mathfrak{F}$.

Remark 1. Theorem 1 can be strengthened—see Theorem 1′.

Theorem 1′. For the metrizability of a separable Lipschitz structure $\mathfrak{F}$, defined on E, it is necessary and sufficient that there exist a convex full set $V \in \mathfrak{F}$ and a monotone set $M \in \mathfrak{F}$ such that each of the sequences $\left\{\frac{1}{2^n}V\right\}$, $\left\{\frac{1}{2^n}M\right\}$ forms a base of the filter $\mathfrak{F}$. Without loss of generality one may assume that M is a full set.

Remark 2. One might suppose that a convex and monotone Lipschitz structure $\mathfrak{F}$ is metrizable if $\mathfrak{F}$ has a countable base. However, this is not so.

The proof of Theorem 1′ is based on the following general method of defining a pseudometric in E. Let $T \subset K(E)$ be an arbitrary full set. Denote by T_n the set $\frac{1}{2^n}T$ (in particular, $T_0 = T$) and define on T the function $D = D_T$: if $\tau \in T_n$ for all n, put $D\tau = 0$; otherwise put $D\tau = \frac{1}{2^n}$, where n is the maximal number for which $\tau \in T_n$. The relation $D\tau \leq \frac{1}{2^n}$ is equivalent to the inclusion $\tau \in T_n$. From $\tau' \prec \tau$ it follows that $D\tau' \leq D\tau$, i.e. the function D is monotone. If $T' \subset T$, then on T' we have $D_{T'} \geq D_T$. It is also clear that on T_n the equality $D_{T_n}\tau = 2^{nD}_T\tau$ holds.

With the aid of the function D_T we define, for some classes, the diameter $d = d_T$. For a simple class τ put

$$d\tau = \inf\left\{\sum D\tau_k\right\},$$

where $\tau_k \in T$ and $\sum \tau_k \succ \tau$. The diameter is certainly defined for a simple $\tau \in T$, and $d\tau \leq D\tau$. Decompose an arbitrary class $\tau \in K(E)$ into simple classes $\tau = \tau_1 + \cdots + \tau_n$ and put $d\tau = \sum d\tau_k$. Thus, the diameter of the class τ is defined if and only if it is defined for all its simple subclasses. If the classes τ_1 and τ_2 have a diameter, then their sum also has a diameter, and

$d(\tau_1 + \tau_2) = d\tau_1 + d\tau_2$. If τ has a diameter and $\tau' \prec \tau$, then τ' also has a diameter, and $d\tau' \leq d\tau$.

We now define the pseudodistance $\rho = \rho_T$. Put $\rho(a, b) = d\{a, b\}$. Generally speaking, ρ is not defined for all pairs of points. If $\rho(a, b)$ exists, we shall write $a\tilde{\rho}b$. The relation $\tilde{\rho}$ is, obviously, reflexive and symmetric. Let $a\tilde{\rho}b$, $b\tilde{\rho}c$. Since $\{a, c\} \prec \{a, b\} + \{b, c\}$, $d\{a, c\}$ exists, i.e. $a\tilde{\rho}c$. Thus $\tilde{\rho}$ is transitive; at the same time we see that $\rho(a, c) \leq \rho(a, b) + \rho(b, c)$. The equivalence relation decomposes E into subsets E_α, $\alpha \in A$, within which there is a pseudometric ρ. The validity of the triangle axiom for ρ has been established, and the fulfillment of the two other axioms of a pseudometric is obvious. One can (in many ways) extend ρ to all of E so that the classes which had no diameter in the pseudometric ρ have, in the new pseudometric ρ', diameter ≥ 1. It is clear that in defining the Lipschitz structure these classes play no role. Therefore one may speak of the Lipschitz structure generated by the pseudometric ρ_T. We shall agree to denote this structure by $\mathfrak{F}_T$.

Let $T' \subset T$. Then, if the diameter $d_{T'}$ is defined, the diameter d_T is also defined, and $d_{T'}\tau \geq d_T\tau$. From this inequality it follows that the pseudometric $\rho_{T'}$ is stronger than ρ_T. Finally, it is easy to verify the equivalence of the pseudometrics ρ_{T_n} and ρ_T.

Theorem 1′ follows from the following lemmas.

Lemma 1. If $\sum D_V\tau_k \leqslant \dfrac{1}{2^{p+1}}$, where $p \geqslant 0$, then $\sum \tau_k \in V_p$, i.e.

$$D_V \sum \tau_k \leqslant \frac{1}{2^p}.$$

Lemma 2. If $\sum \tau_k \in V$, then

$$D_V \sum \tau_k \leqslant 4 \sum D_V\tau_k.$$

Lemma 3. If $d_V\tau < 1/8$, then $\tau \in V$, and moreover,

$$D_V\tau \leqslant 16 d_V\tau.$$

In other words, the structure $\mathfrak{F}_V$ is stronger than the structure $\mathfrak{F}$.

Proof of Lemma 3. Let first τ be a simple class. In this case it is enough to require that $d_V\tau < 1/2$. According to the definition of the diameter d_V, there exist such $\tau_k \in V$ $(k = 1, \dots, n)$ that $\tau \prec \sum \tau_k$, $\sum D_V\tau_k < 1/2$. On the basis of Lemma 1 we have $\sum \tau_k \in V$. Hence $\tau \in V$. Let us now prove that $D_V\tau \leqslant 4d_V\tau$. Suppose the contrary. Then $d_V\tau < \frac{1}{4}D_V\tau$. Hence there exist such $\tau_k \in V$ $(k = 1, \dots, n)$ that $\tau \prec \sum \tau'_k = \tau'$, $\sum D_V\tau_k < \frac{1}{4}D_V\tau$. From this we obtain $\tau' \in V$. By virtue of the monotonicity of the function D_V we have $D_V\tau \leqslant D_V\tau'$. Therefore $4\sum D_V\tau_k < D_V \sum \tau_k$, which contradicts Lemma 2.

Let now τ be an arbitrary class. Decompose it into simple classes

$$\tau = \tau_1 + \dots + \tau_n.$$

Since $d_V\tau_k < 1/2$, it follows that $\tau_k \in V$ $(k = 1, \dots, n)$. From the inequality proved above $D_V\tau_k \leqslant 4d_V\tau_k$ it follows that

$$\sum D_V\tau_k \leqslant 4\sum d_V\tau_k = 4d_V\tau < 1/2.$$

Therefore $\tau \in V$ (Lemma 1). Applying Lemma 2 once more, we obtain

$$D_V\tau \leqslant 4\sum D_V\tau_k \leqslant 16 d_V\tau,$$

which was required to be proved.

Lemma 4. If $\sum \tau_k \in M$, $\sum D_M\tau_k \geqslant \dfrac{1}{2^p}$, then

$$D_M \sum \tau_k \geqslant \frac{1}{2^p}.$$

Lemma 5. If $\sum \tau_k \in M$, then the inequality

$$\sum D_M\tau_k \leqslant 2D_M \sum \tau_k$$

holds.

Lemma 6. The structure $\mathfrak{F}_M$ is weaker than the structure $\mathfrak{F}$.

On the basis of Lemmas 3 and 6 we have $\mathfrak{F}_M \subset \mathfrak{F} \subset \mathfrak{F}_V$. Let us now note that for some m and n we have $M_n \subset V_m \subset M$. Hence we obtain

$$\mathfrak{F}_M \subset \mathfrak{F}_{V_m} \subset \mathfrak{F}_{M_n}.$$

But since $\mathfrak{F}_M = \mathfrak{F}_{M_n}$, $\mathfrak{F}_V = \mathfrak{F}_{V_m}$, it follows that

$$\mathfrak{F}_M = \mathfrak{F}_V = \mathfrak{F}.$$

Theorem 2. A convex and monotone Lipschitz structure $\mathfrak{F}$ is the upper bound of the set of pseudometrizable structures.

Proof. Throughout the proof, M and V denote, respectively, a monotone filled and a convex filled element of the filter $\mathfrak{F}$. In addition, denote by $\mathfrak{F}(T)$, where $T \in \mathfrak{F}$, the filter with basis $\{T_n\}$. It is clear that

$$\mathfrak{F} = \sup_V \mathfrak{F}(V) = \sup_M \mathfrak{F}(M).$$

On the basis of Lemmas 3 and 6 we have $\mathfrak{F}(V) \subset \mathfrak{F}_V$, $\mathfrak{F}_M \subset \mathfrak{F}(M)$. Therefore

$$\sup_V \mathfrak{F}(V) \subset \sup_V \mathfrak{F}_V, \qquad \sup_M \mathfrak{F}_M \subset \sup_M \mathfrak{F}(M),$$

i.e.

$$\sup_M \mathfrak{F}_M \subset \mathfrak{F} \subset \sup_V \mathfrak{F}_V.$$

For every V there exists $M \subset V$. Then we have $\mathfrak{F}_V \subset \mathfrak{F}_M$. Hence it follows that

$$\sup_V \mathfrak{F}_V \subset \sup_M \mathfrak{F}_M.$$

Thus,

$$\mathfrak{F} = \sup_V \mathfrak{F}_V = \sup_M \mathfrak{F}_M.$$

The theorem is proved.

2. Let E be a space with a metric in a semigroup (see (1)). Define on E a Lipschitz structure $\mathfrak{F}$. To this end, denote by F_U, where U is a neighborhood of zero in the metrizing semigroup R, the set of those classes $\tau \in K(E)$ for which $d\tau \in U$ (the diameter of a class is defined exactly as in (2)). The filter with basis $\{F_U\}$ will be denoted by $\mathfrak{F}$. It is easily verified that $\mathfrak{F}$ satisfies the axioms of a Lipschitz structure. The definition of a Lipschitz structure metrizable and pseudometrizable in a semigroup is entirely analogous to Definition 1a.

Theorem 3. For metrizability in a semifield of a separated Lipschitz structure $\mathfrak{F}$, it is necessary and sufficient that $\mathfrak{F}$ be convex and monotone.

To prove necessity, consider in the metrizing semifield R_Δ the set U of functions $f \in \overline{K}_\Delta$ satisfying the inequality $f(e_1) + \dots + f(e_n) \leqslant \varepsilon$, where $e_1, \dots, e_n \in \Delta$, $\varepsilon > 0$. The set U is a neighborhood of zero in $\overline{K}_\Delta$. It is easily proved that F_U is a convex, monotone, and saturated set, and the sets of the indicated form constitute a base of the filter $\mathfrak{F}$.

Sufficiency follows from Theorem 2 and the following lemma.

Lemma 7. The upper bound of a set of Lipschitz structures pseudometrizable in a semifield is a Lipschitz structure pseudometrizable in a semifield.

Proposition 2. If $\mathfrak{F}_k$ $(k = 1, 2, \dots)$ are structures metrizable in the field of real numbers, $\mathfrak{F}_{k+1} \supset \mathfrak{F}_k$, but $\mathfrak{F}_{k+1} \neq \mathfrak{F}_k$, then the structure $\mathfrak{F} = \sup \mathfrak{F}_k$ is not metrizable over the field of real numbers.

Using this proposition, one can construct an example of a convex, monotone, strongly separated structure $\mathfrak{F}$ with a countable base that is not metrizable in the field of real numbers.

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CITED LITERATURE

[1] M. Ya. Antonovskii, V. G. Boltyanskii, T. A. Sarymsakov, *Tr. Tashkentsk. gos. univ.*, no. 191 (1961).
[2] V. Yu. Sandberg, DAN, **145**, No. 2 (1962).

Note: Figure translations are in progress. See original paper for figures.