V. V. KUZNETSOV

1964

SovietRxiv

Abstract

Full Text

V. V. KUZNETSOV

DUALITY OF FUNCTORS IN THE CATEGORY OF SETS WITH A DISTINGUISHED POINT

(Presented by Academician P. S. Aleksandrov, 12 VI 1964)

We consider the category K, whose objects are sets in each of which a certain point is fixed (we shall denote it by 0), and the set $\mathrm{Hom}(X, Y)$ of morphisms of an object X into an object Y is the set of mappings of X into Y taking 0 to 0. Speaking of a functor in K, we shall regard it as covariant and acting from K to K. The equality sign between two functors will denote their isomorphism. The category K is naturally turned into a D-category ([1]), and thereby a duality of functors acting in this category is defined. The functors $H(X, Y)$ and $X \otimes Y$, which occur in the definition of a D-category, are constructed as follows: by $H(X, Y)$ is denoted the set of morphisms of X into Y, with the zero mapping being taken as the distinguished point; by $X \otimes Y$ is denoted the set obtained from the direct product $X \times Y$ by identifying into one point 0 the sets $X \times 0 \cup 0 \times Y$ (here $X, Y \in K$; the values of the functors $H(X, Y)$ and $X \otimes Y$ on morphisms are defined in the natural way). The functors $H(A, X)$ and $A \otimes X$, where A is a fixed object, are denoted respectively by Ω_A and Σ_A.

The main results are Theorems 2 and 3. A proposition analogous to Theorem 3 for the case of the category of topological spaces had already been stated by D. B. Fuks ([2]). However, it still remains unproved.

Let in K there be given a certain family of objects $M_\lambda,\ \lambda \in L$. Considering the direct product $\prod_{\lambda \in L} M_\lambda$ as an object of K, we take as the zero point in it the point with zero components. The set U, obtained from the union of all M_λ by identifying into one zero point the zeros of the sets M_λ, will be called the **bouquet of objects** of the given family and denoted

$$U = \bigvee_{\lambda \in L} M_\lambda.$$

For a given family of functors $P_\sigma,\ \sigma \in S$, the functors $\prod_{\sigma \in S} P_\sigma$ and $\bigvee_{\sigma \in S} P_\sigma$ are naturally defined. We shall call a functor H the **union of the functors** of the given family if for every $X \in K$ we have

$$H(X) = \bigcup_{\sigma \in S} P_\sigma(X)$$

and for every $\sigma \in S$ the functor P_σ is a subfunctor of the functor H.

Let $F = \prod_{\sigma\in S} P_\sigma$. Then one can prove that

$$DF = \bigvee_{\sigma\in S} DP_\sigma$$

(the proof completely coincides with the proof given by D. B. Fuks ([3]) of the analogous fact for topological spaces). This proposition means that if φ_σ is the natural mapping $P_\sigma \to \Sigma_B$, and f is an arbitrary mapping $F \to \Sigma_B$, then $f\varphi_\sigma$ is nonzero for at most one $\sigma \in S$; on the other hand, for any $\sigma_0 \in S$ and $\psi : P_{\sigma_0} \to \Sigma_B$ there exists a unique $f : F \to \Sigma_B$ such that $f\varphi_{\sigma_0} = \psi$, while for $\sigma \neq \sigma_0$ the mapping $f\varphi_\sigma$ is zero.

Let L and S be two sets, and suppose that to each $\lambda \in L$ there is assigned a set $S_\lambda \subset S$, so that $S = \bigcup_{\lambda\in L} S_\lambda$, while to each $\sigma \in S$ there corresponds some functor G_σ. We shall consider the functor

$$F = \bigcup_{\lambda\in L} F_\lambda,$$

where

$$F_\lambda = \prod_{\sigma\in S_\lambda} G_\sigma.$$

Here, for every $X \in K$, in $F_{\lambda'}(X)$ and $F_{\lambda''}(X)$ we identify those two points $\{g'_\sigma\}$, $\sigma \in S_{\lambda'}$, and $\{g''_\sigma\}$, $\sigma \in S_{\lambda''}$, for which, if $\sigma \in S_{\lambda'} \cap S_{\lambda''}$, we have $g'_\sigma = g''_\sigma$, while, if $\sigma \notin S_{\lambda''}$ and $\sigma \notin S_{\lambda'}$, respectively, $g'_\sigma = 0$ and $g''_\sigma = 0$. It turns out that the functor DF can be obtained by means of a construction of the same kind, the role of the sets S and S_λ being played respectively by the set S and by the collections s of points of the set S which meet each S_λ in not more than one point, and the role of the set L by the family R of all such collections. Namely, we have

Lemma 1. *The functor DF is isomorphic to the functor*

$$T = \bigcup_{s\in R} \prod_{\sigma\in S} DG_\sigma.$$

Let $f : F \to \Sigma_B$ be an arbitrary point of $DF(B)$, and let f_λ be the mapping $F_\lambda \to \Sigma_B$ induced by the mapping f; let ψ_B be the mapping of the functor $G_\sigma \subset F_\lambda$ ($\sigma \in S_\lambda$) into Σ_B induced by f_λ (it is not hard to see that in the case where $\sigma \in S_{\lambda'}$ and $\sigma \in S_{\lambda''}$, the mappings $\varphi_{\lambda'}$ and $\varphi_{\lambda''}$ give rise to one and the same mapping ψ_σ). Then the set s of those indices σ for which $\psi_\sigma \neq 0$ belongs to the family R. Indeed, if there existed $\sigma', \sigma'' \in s$ and $\lambda \in L$ such that $\sigma', \sigma'' \in S_\lambda$, then the mapping φ_λ would induce nonzero mappings $\psi_{\sigma'}$ and $\psi_{\sigma''}$ for two factors $G_{\sigma'}$ and $G_{\sigma''}$ of the direct product F_λ, which is impossible.

Thus the collection $\bar{f} = \{\psi_\sigma\}$, $\sigma \in s$, may be regarded as a point of $T(B)$. The mapping

$$\chi_B : DF(B) \to T(B),$$

under which $f \mapsto \bar{f}$, is obviously a monomorphism. We shall show that χ_B is also an epimorphism.

Let $s \in R$, and let

$$f \in \prod_{\sigma \in S} DG_\sigma(B)$$

be a collection of mappings $\psi_\sigma : G_\sigma \to \Sigma_B$. For each $\lambda \in L$ define a mapping $\varphi_\lambda : F_\lambda \to \Sigma_B$ by the following rule: if S_λ contains a point $\sigma \in s$, then φ_λ is defined by the mapping ψ_B, so that $\varphi_\lambda \theta_\sigma = \psi_\sigma$, where θ_σ is the natural mapping $G_\sigma \to F_\lambda$; if, however, the set $S_\lambda \cap s$ is empty, then the mapping φ_λ is zero. It is obvious that, for any $\lambda', \lambda'' \in L$ and any $X \in K$, the mappings $(\varphi_{\lambda'})_X$ and $(\varphi_{\lambda''})_X$ coincide on the set $F_{\lambda'}(X) \cap F_{\lambda''}(X)$, and this means that the mappings φ_λ define a mapping $\bar{f} : F \to \Sigma_B$. Clearly $\chi_B(\bar{f}) = \bar{f}$. The functoriality of the mappings χ_B is easily verified. The lemma is proved.

Choose in K an object M, and specify in it some family $\mathfrak{M}$ of subobjects M_λ, $\lambda \in L$. Define the functor $\Omega_M^{\mathfrak{M}}$ as a subfunctor of the functor Ω_M, for which $\Omega_M^{\mathfrak{M}}(X)$ is the set of those mappings $f : M \to X$ such that $f(M_\lambda) = 0$ for at least one $\lambda \in L$. Associate to every $\Lambda \subset L$ the set $N_\Lambda \subset M$, putting

$$N_\Lambda = \left(\bigcap_{\lambda \in \Lambda} M_\lambda \setminus \bigcup_{\lambda \in L \setminus \Lambda} M_\lambda \right) \cup 0,$$

if Λ is nonempty, and

$$N_\Lambda = \left(M \setminus \bigcup_{\lambda \in L} M_\lambda \right) \cup 0,$$

if Λ is empty. Obviously,

$$M/M_\lambda = \bigvee_{\lambda \in L,\ \lambda \notin \Lambda} N_\Lambda.$$

Then we have a series of isomorphisms:

$$\Omega_M^{\mathfrak{M}} = \bigcup_{\lambda \in L} \Omega_{M/M_\lambda} = \bigcup_{\lambda \in L} \prod_{\Lambda \subset L - \lambda} \Omega_{N_\Lambda}.$$

We see that the functor $\Omega_M^{\mathfrak{M}}$ belongs to the class of functors whose duals were found in Lemma 1.

Using Lemma 1 and taking into account that $D\Omega_A = \Sigma_A$, $D\Sigma_A = \Omega_A$, one can verify the validity of the following three assertions.

Lemma 2.

$$D\Omega_M^{\mathfrak{M}} = \bigcup_{s \in S} \prod_{\Lambda \in s} \Sigma_{N_\Lambda \setminus \Lambda},$$

where S is the family of all collections s of pairwise disjoint nonempty subsets $\Lambda \subset L$.

Lemma 3.

$$DD\Omega_M^{\mathfrak{M}} = \bigcup_{r \in R} \prod_{\Lambda \in r} \Omega_{N_L \setminus \Lambda},$$

where R is the family of all sets r of pairwise intersecting nonempty $\Lambda \subset L$.

Theorem 1. Every functor of the form $D\Omega_M^{\mathfrak{M}}$ is reflexive.

Let now F be some functor. Denote by I the two-point object of K, and for each $\alpha \in DF(I)$ construct in $A = F(I)$ an equivalence λ_α as the least one for which, for any $X \in K$ and $y \in F(X)$, if $\mu', \mu'' \in \mathrm{Hom}(X, I)$ are such that

$$\mu' \alpha_X(y) = \mu'' \alpha_X(y) \neq 0,$$

then

$$F(\mu')y \sim F(\mu'')y;$$

and if $\mu \in \mathrm{Hom}(X, I)$ is such that $\mu \alpha_X(y) = 0$, then $F(\mu)y$ is equivalent to 0. We shall denote the quotient set of the set A by the equivalence λ_α (in which the class of points equivalent to zero is distinguished) by A_α, and the identification map $A \to A_\alpha$ by π_α. We also note that, for each $B \in K$, there is a morphism $\varepsilon_B : B \to I$ such that $\varepsilon_B(b) \neq 0$ for any nonzero point $b \in B$, and, for any $\varphi \in DF(B)$, a map $\alpha^\varphi \in DF(I)$ defined by the formula

$$\alpha^\varphi = DF(\varepsilon_B)\varphi.$$

Lemma 4. Let $\varphi \in DF(B)$. If

$$a' \sim a'' \pmod{\lambda_{\alpha^\varphi}},$$

then

$$\varphi_I(a') = \varphi_I(a'').$$

Conversely, if for some $\alpha \in DF(I)$ and $\psi : A \to B$, from

$$a' \sim a'' \pmod{\lambda_\varphi}$$

it follows that

$$\psi(a') = \psi(a''),$$

then there exists a $\varphi \in DF(B)$ such that $\psi = \varphi_I$.

It suffices to prove the first part of the lemma for the case when there exist $X \in K$, $y \in F(X)$, and $\mu', \mu'' \in \mathrm{Hom}(X, Y)$ such that

$$\mu'(\alpha_X^\varphi(y)) = \mu''(\alpha_X^\varphi(y)) \neq 0$$

and

$$F(\mu')y = a', \qquad F(\mu'')y = a''.$$

But then

$$\varphi_I(a') = \Sigma_B(\mu')\varphi_X(y), \qquad \varphi_I(a'') = \Sigma_B(\mu'')\varphi_X(y),$$

and, since for some $b \in B$ we have

$$\varphi_X(y) = \alpha_X^{\varphi}(y) \otimes b,$$

it follows that

$$\varphi_I(a') = \varphi_I(a'') = b.$$

Being under the hypotheses of the second part of the lemma, for any $X \in K$ and $y \in F(X)$ put

$$\varphi_X(y) = 0$$

if

$$\alpha_X(y) = 0;$$

but if

$$\alpha_X(y) \neq 0,$$

then

$$\varphi_X(y) = \alpha_X(y) \otimes b,$$

where $b = F(\mu)y$ for such a $\mu : X \to I$ that

$$\mu(\alpha_X(y)) \neq 0.$$

Let us verify that the morphisms

$$\varphi_X : F(X) \to B \otimes X$$

define a map of the functor F into the functor Σ_B. Suppose that for some

$$\nu : X' \to X''$$

we have

$$F(\nu)y' = y''.$$

Compare the points

$$x' \otimes b' = \Sigma_B(\nu)\varphi_{X'}(y')$$

and

$$x'' \otimes b'' = \varphi_{X''}(y'').$$

Since

$$\nu(\alpha_{X'}(y')) = \alpha_{X''}(y''),$$

we have

$$x' = x'' = \alpha_{X''}(y''),$$

and if

$$\alpha_{X''}(y'') \neq 0,$$

then there exist $\mu' : X' \to I$ and $\mu'' : X'' \to I$ such that

$$\mu'(\alpha_{X'}(y')) = \mu''(\alpha_{X''}(y'')) \neq 0$$

and

$$F(\mu')y' = F(\mu'')y'',$$

and therefore

$$b' = b''.$$

Theorem 2. For every functor F, the functor DF has the form $\Omega_M^{\mathfrak{M}}$.

We shall exhibit an isomorphism of the functors DF and $\Omega_M^{\mathfrak{M}}$ for the set M obtained from the bouquet

$$\bigvee_{\alpha \in DF(I)} A_\alpha$$

by identifying, for any $\alpha', \alpha'' \in DF(I)$, points $a' \in A_{\alpha'}$ and $a'' \in A_{\alpha''}$ if

$$\pi_{\alpha'}^{-1}(a') = \pi_{\alpha''}^{-1}(a''),$$

and for the family $\mathfrak{M}$ of sets M_α, $\alpha \in DF(I)$, defined by the formula

$$M_\alpha = (M \setminus \varkappa(A_\alpha)) \cup 0,$$

where $\varkappa$ is the natural map

$$\bigvee_{\alpha \in DF(I)} A_\alpha \to M.$$

For each $B \in K$ construct a map

$$\chi_B : DF(B) \to \Omega_M^{\mathfrak{M}}(B)$$

by assigning to each $\varphi \in DF(B)$ the map $\omega : M \to B$ that is zero on the set M_{α^φ} and equal to

$$\theta_\varphi \varkappa^{-1}$$

on the set

$$M \setminus M_{\alpha^\varphi},$$

where θ_φ is the map

$$A_{\alpha^\varphi} \to B$$

induced by the map φ_I, so that

$$\varphi_I = \theta_\varphi \pi_{\alpha^\varphi}$$

(the existence of θ_φ is ensured by Lemma 4).

For any B, the map χ_B is a monomorphism. Indeed, suppose that for distinct $\varphi', \varphi'' \in DF(B)$ we have

$$\chi_B(\varphi') = \chi_B(\varphi'') = \omega.$$

This means that

$$\omega(M_{\alpha^{\varphi'}}) = \omega(M_{\alpha^{\varphi''}}) = 0,$$

i.e., for all $a' \in A_{\alpha^{\varphi'}}$ for which

$$\varkappa(a') \notin \varkappa(A_{\alpha^{\varphi''}}),$$

we have

$$\omega(\varkappa(a')) = 0.$$

Hence the map φ'_I differs from the zero map only at those nonzero points $a \in A$ for which

$$\varkappa(\pi_{\alpha^{\varphi'}}(a')) = \varkappa(\pi_{\alpha^{\varphi''}}(a''));$$

it is easy to conclude that

$$\varphi'_I = \varphi''_I.$$

But it is not difficult to verify directly that, for distinct $\varphi', \varphi'' \in DF(B)$, the maps φ'_I and φ''_I are distinct.

On the other hand, let us show that for every B the mapping χ_B is also an epimorphism. For this, for an arbitrary $\omega \in \Omega_M^{\mathfrak{M}}(B)$ choose α for which $\omega(M_\alpha) = 0$, and, by means of the mapping $\psi = \omega\chi\pi_\alpha$, which obviously satisfies the condition of the second part of Lemma 4, construct $\varphi : F \to \Sigma_B$ for which $\varphi_I = \psi$. We shall verify that $\chi_B(\varphi) = \omega$. Denoting by A_0 the set of those points $a \in A$ for which $\omega(\chi(\pi_\alpha(a))) = 0$, we note that if $a \in A$, $X \in K$, $y \in F(X)$, and $\mu : X \to I$ are such that $F(\mu)y = a$, then $\alpha_X(y) = \alpha_x(y) \neq 0$ when $a \notin A_0$, and $\alpha_X^\varphi(y) = 0$ for $a \in A_0$. Hence it is easy to conclude that $\pi_\alpha^{-1}(0) = A_0$ and that on the set $A \setminus A_0$ the equivalences λ_a and $\lambda_{a\varphi}$ coincide. But then $M_{\alpha\varphi} = M_\alpha \setminus \chi(\pi_\alpha(A_0))$. Comparing the mappings ω and $\omega' = \chi_B(\varphi)$, we shall have $\omega'(M_{\alpha\varphi}) = \omega(M_{\alpha\varphi}) = 0$ and $\omega'(M \setminus M_{\alpha\varphi}) = \theta_\varphi(\chi^{-1}(M \setminus M_{\alpha\varphi})) = \omega(M \setminus M_{\alpha\varphi})$, i.e. $\omega' = \omega$.

Thus the mappings $\chi_B : DF(B) \to \Omega_M^{\mathfrak{M}}(B)$ are isomorphisms. It is easy to verify the functoriality of these mappings. The theorem is proved.

Theorem 3. *For every functor F in the category K, the functor DF is reflexive.*

From Theorems 1 and 2 it follows immediately that DDF is reflexive for an arbitrary functor F. It remains to refer to (3), where D. B. Fuks proved that from the reflexivity of DDF follows the reflexivity of the functor DF. (D. B. Fuks' s proof was given for the category of topological spaces, but in substance it does not use the specifics of this category.)

I take this opportunity to express my gratitude to A. S. Schwarz for advice and suggestions.

Received 22 V 1964

REFERENCES

1. A. S. Schwarz, DAN, **148**, No. 2, 288 (1963).

2. D. B. Fuks, DAN, **141**, No. 4, 818 (1961).

3. D. B. Fuks, Dissertation, M. V. Lomonosov Moscow State University, 1963.

Note: Figure translations are in progress. See original paper for figures.