ON THE CONVERGENCE OF THE PROJECTION METHOD IN EIGENVALUE PROBLEMS

1964

SovietRxiv

Abstract

Full Text

MATHEMATICS

V. A. MEDVEDEV

ON THE CONVERGENCE OF THE PROJECTION METHOD IN EIGENVALUE PROBLEMS

(Presented by Academician G. I. Petrov on 27 XII 1963)

In [1] the convergence of the projection method for linear inhomogeneous equations and eigenvalue problems was studied. The convergence of approximate eigenvectors was not considered. The present article is devoted to this question.

Let a linear bounded operator $L(\lambda)$ be given, defined on a Hilbert space H_1 and with range in a Hilbert space H_2, where $L(\lambda)$ is a holomorphic function of the parameter λ from some domain D of the complex plane. It is required to find the eigenvalues and eigenvectors of the equation

$$L(\lambda)x = 0. \tag{1}$$

Take a projection-complete in H_1 [2] sequence of finite-dimensional subspaces $\{R_n\}$ and a projection-complete in H_2 sequence of subspaces $\{M_n\}$ such that $\dim M_n = \dim R_n$. Denote by Q_n the operator of orthogonal projection onto the subspace R_n, and by P_n the operator of orthogonal projection onto the subspace M_n. We seek approximate eigenvalues and eigenvectors as the eigenvalues and eigenvectors of the equation

$$P_n L(\lambda)x_n = 0, \tag{2}$$

where the domain of definition of the operator $P_n L(\lambda)$ is the subspace R_n. Let λ_0 be an eigenvalue of equation (1), and let $\{\lambda_n\}$ be a sequence of eigenvalues of equations (2) such that $\lim \lambda_n = \lambda_0$. We shall denote by x_n the eigenvector of equation (2) belonging to the eigenvalue λ_n, and assume $|x_n| = 1$. Denote by S the operator of orthogonal projection onto the eigensubspace of equation (1) belonging to the eigenvalue λ_0. Next denote $\Pi = E - S$. Then x_n can be written in the form

$$x_n = Sx_n + \Pi x_n.$$

Πx_n characterizes the deviation of the vector x_n from the eigensubspace of equation (1); therefore it is necessary to determine the behavior of this term as $n \to \infty$.

Theorem 1. The sequence $\{\Pi x_n\}$ converges weakly to zero.

Proof. We have

$$P_n L(\lambda_n) x = 0. \tag{3}$$

Hence, using the continuous dependence of the operator L on λ, we obtain $\lim_{n\to\infty} P_n L(\lambda_0) x_n = 0$, or

$$\lim_{n\to\infty} P_n L(\lambda_0) \Pi x_n = 0. \tag{4}$$

For $z \in H_2$ we have

$$(\Pi x_n, L^*(\lambda_0) z) = (L(\lambda_0)\Pi x_n, z) =$$

$$= (L(\lambda_0)\Pi x_n, P_n z) + (L(\lambda_0)\Pi x_n, z - P_n z) =$$

$$= (P_n L(\lambda_0)\Pi x_n, z) + (L(\lambda_0)\Pi x_n, z - P_n z).$$

By virtue of equality (4) and the projective completeness of the sequence $\{M_n\}$, we obtain $\lim(\Pi x_n, L^*(\lambda_0)z) = 0$, whence follows the weak convergence of the sequence $\{\Pi x_n\}$ to zero, since the set of vectors of the form $L^*(\lambda_0)z$ is dense in the subspace that is the orthogonal complement of the eigenspace of equation (1) corresponding to the eigenvalue λ_0.

Denote

$$\mu_n = \min_{z_n \in R_n} \frac{|P_{nLz} n|}{|z_n|}.$$

We shall say that, for the operator L, condition (A') is satisfied if

$$\mu = \lim_{n\to\infty} \mu_n > 0.$$

(Obviously, if N. I. Pol' skii' s condition (A) [(2)] is satisfied and there exists a bounded operator L^{-1}, then condition (A') is also satisfied. If the operator L is bounded, then, when condition (A') is satisfied, condition (A) is satisfied. In [(1)], in all the cases considered, condition (A') was in fact used; this means that the norm of the operator $(P_{nL})^{-1}$, starting from some n, is bounded uniformly with respect to n.)

Theorem 2. *Let the operator $L(\lambda_0)$ have the form $L(\lambda_0) = L_0 + T$, where T is a completely continuous operator, and suppose that condition (A') is satisfied for the operator L_0. Then $\lim_{n\to\infty} \Pi x_n = 0$.*

Proof. First we show that the eigenspace of equation (1) corresponding to the eigenvalue λ_0 is finite-dimensional. It is easy to show that from the fulfillment of condition (A') for L_0 it follows that

$$\inf_{x\in H_1} \frac{|L_0 x|}{|x|} > 0. \tag{5}$$

Suppose that there exists an infinite sequence $\{x^{(m)}\}$ $(m = 1, 2, ...)$ of eigenvectors of equation (1) for $\lambda = \lambda_0$. Obviously, it may be assumed orthonormal and, consequently, weakly convergent to zero. Then $\lim_{m\to\infty} Tx^{(m)} = 0$, and therefore also $\lim_{m\to\infty} L_0 x^{(m)} = 0$, which contradicts inequality (5). Thus the eigenspace is finite-dimensional. It follows from this that the operator S is completely continuous.

Assume now that Theorem 2 is false. Then there is a number $\varepsilon > 0$ and a subsequence of indices n for which the inequality

$$|\Pi x_n| > \varepsilon \tag{6}$$

holds.

In what follows we shall consider only such n. Represent x_n in the form

$$x_n = Q_{nSx}n + (E - Q_n)Sx_n + \Pi x_n.$$

Denote

$$y_n = (E - Q_n)Sx_n + \Pi x_n.$$

Obviously, $y_n \in R_n$. From the complete continuity of the operator S it follows that

$$\lim_{n\to\infty} \|(E - Q_n)S\| = 0 \tag{7}$$

and, hence, the sequence $\{y_n\}$, like $\{\Pi x_n\}$, converges weakly to zero. From inequality (6) and equality (7) it follows that $|y_n| > \varepsilon/2$ for sufficiently large n. Using equalities (4) and (7), we obtain

$$\lim_{n\to\infty} \frac{|P_{nL}(\lambda_0)y_n|}{|y_n|} = 0, \tag{8}$$

whence

$$\lim_{n\to\infty} \frac{|P_n L_0 y_n|}{|y_n|} = 0,$$

since $\lim_{n\to\infty} Ty_n = 0$. We have arrived at a contradiction with the fact that condition (A') is satisfied for the operator L_0. The theorem is proved.

Let now $H_1 = H_2 = H$ and $M_n = R_n$.

Theorem 3. *Suppose that for every sequence $z_n \in H$ weakly converging to zero, $|z_n| = 1$ $(n = 1, 2, ...)$, the inequality*

$$\lim_{n\to\infty} |(L(\lambda_0)z_n, z_n)| > 0$$

is satisfied. Then $\lim_{n\to\infty} \Pi x_n = 0$.

Proof. The finite dimensionality of the eigenspace of equation (1) can be proved analogously to how this was done in the proof of Theorem 2. Suppose that Theorem 3 is false. Then, just as in the proof of Theorem 2, we obtain a sequence $\{y_n \in R_n\}$ weakly converging to zero such that equality (8) is satisfied, and moreover $|y_n| > \varepsilon/2$ for sufficiently large n. Then the sequence of vectors $z_n = y_n/|y_n|$ weakly converges to zero, $|z_n| = 1$, and the equality

$$\lim_{n\to\infty} |P_n L(\lambda_0)z_n| = 0$$

is valid. Since

$$|(L(\lambda_0)z_n, z_n)| \leqslant |P_n L(\lambda_0)z_n|,$$

we have

$$\lim_{n\to\infty} |(L(\lambda_0)z_n, z_n)| = 0,$$

which contradicts the conditions of the theorem.

When the multiplicity of the eigenvalue λ_0 of equation (1) is equal to one, Theorems 2 and 3 completely resolve the question of the convergence of approximate eigenvectors for the operators under consideration. If the multiplicity of the eigenvalue λ_0 is greater than one, the following consequences follow from them.

Corollary 1. *Every strongly convergent subsequence of the sequence* $\{x_n\}$ *of eigenvectors of equations (3) converges to an eigenvector of equation (1) belonging to the eigenvalue* λ_0*, and at least one such subsequence exists.*

Corollary 2. *The dimension of the eigenspace of equation (3) belonging to the eigenvalue* λ_n*, for sufficiently large* n*, does not exceed the dimension of the eigenspace of equation (1) belonging to the eigenvalue* λ_0*.*

As the example of N. I. Pol' skii ([3]) shows, it may happen that a certain eigenvector cannot be obtained as the limit of a sequence of approximate eigenvectors or of their linear combinations.

Scientific Research Institute of Mechanics
Moscow State University
named after M. V. Lomonosov

Received
19 XII 1963

References

1. V. A. Medvedev, *Prikl. matem. i mekh.*, **27**, 6, 1148 (1963).
2. N. I. Pol' skii, *DAN*, **143**, No. 4, 787 (1962).
3. N. I. Pol' skii, *Ukr. matem. zhurn.*, **7**, 1 (1955).

Note: Figure translations are in progress. See original paper for figures.