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Abstract

Full Text

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ON ONE PARTICULAR CASE OF THE OBLIQUE DERIVATIVE PROBLEM FOR HARMONIC FUNCTIONS IN THREE-DIMENSIONAL DOMAINS

In a simply connected finite domain D of the variables x, y, z, bounded by a Lyapunov surface S, one seeks a regular harmonic function $U(x, y, z)$, continuous together with its first partial derivatives in $D + S$ and satisfying the boundary condition

$$\mathbf{P}(p, q, r) \cdot \operatorname{grad} U = f, \qquad (x, y, z) \in S, \tag{1}$$

where $p = x - a,\ q = y,\ r = z$, and $a = \text{const}$.

In view of the fact that, along with $U(x, y, z)$, the expression $(x-a)U_x+yU_y+zU_z$ is a regular harmonic function in the domain D, from (1) we have

$$(x-a)U_x + yU_y + zU_z = V(x, y, z), \qquad (x, y, z) \in D, \tag{2}$$

where $V(x, y, z)$ is a known regular harmonic function in D, coinciding with f on S.

Thus, the problem posed above (which in what follows we shall call problem (1)) is reduced to finding regular harmonic solutions in the domain D of the linear first-order equation (2).

For simplicity, below we shall restrict ourselves to considering the case of the ball D with boundary $S : x^2 + y^2 + z^2 = 1$. In this case the function $V(x, y, z)$ is given by the well-known Poisson formula.

First suppose that $|a| < 1$. From the form of equation (2) it is clear that, for the existence of a solution of problem (1), it is necessary that the function f satisfy the integral condition

$$V(a, 0, 0) = \frac{1}{4\pi} \iint_S \frac{1 - a^2}{(1 + a^2 - 2a\cos\theta)^{3/2}} f\, dS = 0. \tag{3}$$

If condition (3) is satisfied, then a regular harmonic solution of equation (2) exists and is given by the formula

$$U(x,y,z) = \int_0^1 V[a + t(x-a), ty, tz]\, t^{-1} dt + C,$$

where C is an arbitrary constant. In the case under consideration, problem (1) has no other solutions ([1], [2]).

Thus, for $|a| < 1$, condition (3) is necessary and sufficient for the existence of a solution of problem (1).

Let now $|a| > 1$. In this case all solutions of equation (2) that are twice continuously differentiable in the ball D are given by the formula

$$U(x,y,z) = \int_{\frac{a}{a-x}}^1 V[a + t(x-a), ty, tz]\, t^{-1} dt + \varphi(\xi, \eta), \tag{4}$$

where $\varphi(\xi, \eta)$ is a twice continuously differentiable function of the variables

$$\xi = \frac{y}{x-a}, \qquad \eta = \frac{z}{x-a}.$$

In order that the function $U(x,y,z)$ represented by formula (4) be a regular solution of Laplace' s equation in the ball D, it is necessary and sufficient that the function $\varphi(\xi, \eta)$ satisfy the linear elliptic equation of the second order

$$(1+\xi^2)\varphi_{\xi\xi} + (1+\eta^2)\varphi_{\eta\eta} + 2\xi\eta\varphi_{\xi\eta} + 2\xi\varphi_\xi + 2\eta\varphi_\eta = -\frac{\partial}{\partial\alpha}(1-\alpha)V[\alpha a, (\alpha-1)a\xi, (\alpha-1)a\eta]\Big|_{\alpha=0}. \tag{5}$$

Let us note that the direction $\mathbf{P}(x-a, y, z)$ goes out into the plane tangent to the sphere S along the circle $x = \frac{1}{a}$, $x^2 + y^2 + z^2 = 1$. At the same time, when the point (ξ, η) runs over the disk $\xi^2 + \eta^2 \le \frac{1}{a^2-1}$, the point (x,y,z) runs over the ball $x^2 + y^2 + z^2 \le 1$.

Thus, for $|a| > 1$, problem (1) always has solutions, which can be represented in the form $U = U_0 + \Psi$, where U_0 is a particular solution of the nonhomogeneous equation (5), and $\Psi(\xi, \eta)$ is the general regular solution of the homogeneous equation

$$(1+\xi^2)\Psi_{\xi\xi} + (1+\eta^2)\Psi_{\eta\eta} + 2\xi\eta\Psi_{\xi\eta} + 2\xi\Psi_\xi + 2\eta\Psi_\eta = 0. \tag{6}$$

The formula $U = U_0 + \Psi$ gives all solutions of problem (1). This follows from the fact that equation (6) has a unique solution, regular in the disk $\xi^2 + \eta^2 < \frac{1}{a^2-1}$, assuming prescribed continuous values on the circle $\xi^2 + \eta^2 = \frac{1}{a^2-1}$, or, what is the same, on the circle $x^2 + y^2 + z^2 = 1$, $x = \frac{1}{a}$ (see ([12])).

Consequently, for $|a| > 1$ a solution of problem (1) always exists, and it is determined uniquely if its continuous values are prescribed in advance on the set of points of the sphere S at which the direction of the vector $\mathbf{P}(x-a, y, z)$ goes out into the plane tangent to S.

Problem (1) is investigated analogously also in the case when

$$p = p_0 + ax + by + cz, \qquad q = q_0 - bx + ay + dz, \qquad r = r_0 - cx - dy + az.$$

It should be noted that, in the case considered above, the Kronecker index characterizing the rotation of the vector field $\mathbf{P}(x-a, y, z)$ prescribed on S is equal to $+1$ or 0 according as $|a| < 1$ or $|a| > 1$. In the case $|a| = 1$, although the concept of the Kronecker index loses its meaning, problem (1) is investigated in exactly the same way as in the case $|a| < 1$.

The degree of overdetermination or underdetermination of the oblique derivative problem (1) for $\mathbf{P} \neq 0$ is most closely connected with the Kronecker index of the vector field $\mathbf{P}(p, q, r)$ on the surface S. We shall return to this question in another note.

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References

[1] A. V. Bitsadze, DAN, 148, No. 4 (1963). [2] A. V. Bitsadze, *Outlines of the Joint Soviet-American Symposium on Partial Differential Equations*, 1963, Novosibirsk, p. 46.

Note: Figure translations are in progress. See original paper for figures.