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The question of the existence and uniqueness of a solution of the Hammerstein integral equation

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Abstract

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ON SOLUTIONS OF HAMMERSTEIN INTEGRAL EQUATIONS

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The question of the existence and uniqueness of a solution of the Hammerstein integral equation

$$u(s) = \int_D K(s,t) f(t, u(t))\, dt \tag{1}$$

has been considered by many authors (see, for example, ($^{1-6}$)). I have also obtained some results in this direction (^{7}). In doing so it was most often assumed that the kernel $K(s,t)$ is positive definite, quasi-positive, or quasi-negative. The general case, apparently, has not yet been systematically studied. M. M. Vainberg and R. I. Kachurovskii (^{5}) obtained some interesting results for the general case. In the present article the author also attempts to consider the general case.

For the sake of generality, instead of equation (1) we consider the system

$$u_i(s) = K_i f_i u(s) \qquad (i = 1, 2, \dots, m), \tag{2}$$

where

$$K_i f_i u(s) = \int_D K_i(s,t) f_i(t, u_1(t), \dots, u_m(t))\, dt.$$

1. Special case.

Let D be a measurable set of finite-dimensional Euclidean space and $mD < \infty$. Let $M(u)$, $N(v)$ $(-\infty < u, v < \infty)$ be mutually complementary N-functions, the second of which satisfies the Δ_2-condition. Let L_M^*, L_N^* be Orlicz spaces of functions defined on D, satisfying the relation $E_M \subset L_2 \subset L_N^*$.

Suppose that the functions $f_i(t, u_1, \dots, u_m)$ $(i = 1, 2, \dots, m;\ t \in D,\ -\infty < u_j < \infty\ (j = 1, \dots, m))$ satisfy the following conditions:

δ_1) the Carathéodory condition, i.e. they are measurable in t for each $(u_1, \dots, u_m)$ and continuous in $(u_1, \dots, u_m)$ for almost all t.

δ_2) There exists a function $F(t, u_1, \dots, u_m)$, satisfying the Carathéodory condition, such that $F(t, 0, \dots, 0) \in L$ and

$$f_i(t, u_1, \dots, u_m) = \frac{\partial}{\partial u_i} F(t, u_1, \dots, u_m) \qquad (j = 1, 2, \dots, m).$$

δ_3) The operator

$$f(u) = (f_1(u), \dots, f_m(u)),$$

where

$$f_i(u) = f_i(t, u_1(t), \dots, u_m(t)),$$

acts continuously from

$$\mathcal{E}_M = \underbrace{E_M + \dots + E_M}_{m} \quad \text{to} \quad \mathcal{L}_N^* = \underbrace{L_N^* + \dots + L_N^*}_{m}.$$

Suppose also that the functions $K_i(s, t)$ are defined on $D \times D$ and generate bounded linear integral operators K_i, acting from L_N^* into E_M. Let these operators, considered on L_2, be self-adjoint operators, and let the positive part of the spectrum of each operator K_i have positive infimum λ_i.

Put

$$T_i = (b_{iK} i - I)^{-1} K_i,$$

i.e.

$$T_i = \int_{m_i}^{M_i} \frac{\lambda}{b_i \lambda - 1} \, dE_\lambda^{(i)} = \int_{m_i}^{0} \frac{\lambda}{b_i \lambda - 1} \, dE_\lambda^{(i)} + \int_{\lambda_i - \theta}^{M_i} \frac{\lambda}{b_i \lambda - 1} \, dE_\lambda^{(i)},$$

where $E_\lambda^{(i)}$ is the resolution of the identity for the operator K_i, and $b_i \lambda_i > 1$. Therefore both operators on the right-hand side of the last equality are bounded and positive definite. It follows that the operator T_i is the same. Moreover, it can be shown that T_i is bounded as an operator from L_N^* into E_M.

Theorem 1. Let $f_i(t, u_1, \dots, u_m)$ and $K_i(s, t)$ $(i = 1, 2, \dots, m)$ satisfy the conditions stated above. If

$$F(t,u_1{+}h_1,\dots,u_m{+}h_m){-}F(t,u_1,\dots,u_m){-}\sum_{i=1}^{m} f_i(t,u_1,\dots,u_m)h_i \ge \frac{1}{2}\sum_{i=1}^{m}(b_i{-}c_i)h_i^2, \tag{I}$$

where $b_i\lambda_i > 1,\ c_i\|T_i\| < 1$, then there exists one and only one solution of system (2).

For the proof, system (2) is replaced by the equivalent system

$$u_i = T_i g_i u \qquad (i=1,\dots,m,\ u=(u_1,\dots,u_m)), \tag{3}$$

where

$$g_i(t,u_1,\dots,u_m) = b_i u_i - f_i(t,u_1,\dots,u_m).$$

Define the operator

$$Hu = (H_1 u_1,\dots,H_m u_m),$$

where

$$H_i = T_i^{1/2} \qquad (i=1,2,\dots,m),$$

and the functional

$$G(u) = \int_D G(t,u_1(t),\dots,u_m(t))\,dt,$$

where

$$G(t,u_1,\dots,u_m) = \frac{1}{2}\sum_{i=1}^{m} b_i u_i^2 - F(t,u_1,\dots,u_m).$$

Then the functional defined in L_2,

$$\varphi(u) = \frac{1}{2}(u,u) - G(Hu),$$

as is easily verified, satisfies the relation

$$\varphi(u+h)-\varphi(u)-(\varphi'(u),h) \ge \frac{1}{2}(1-q)\|h\|^2,$$

where $q = \max_i c_i \|T_i\| < 1$. From the results of $(^{2,8})$ it follows that $\varphi(u)$ is a convex, weakly lower semicontinuous functional having in L_2 at least one minimum point u_0. Then Hu is a solution of system (3). It is easy to show that this solution is unique.

Remark 1. Condition (I) can be replaced by the condition

$$\sum_{i=1}^{m} \left[f_i(t, u_1 + h_1, \dots, u_m + h_m) - f_i(t, u_1, \dots, u_m)\right] h_i \geq \sum_{i=1}^{m} (b_i - c_i) h_i^2. \qquad \text{(I')}$$

Remark 2. It can be shown that every sequence minimizing the functional $\varphi(u)$ converges to u_0 in L_2.

Remark 3. Theorem 1 can be formulated in a more general form. For simplicity we restrict ourselves to the case of a Hilbert space.

Theorem $1'$**.** Let K be a bounded self-adjoint operator acting in a Hilbert space H. Let the positive part

of the spectrum of the operator K has a positive infimum λ_1. Let $f(u)$ be a discontinuous potential operator acting in H: $f(u) = \operatorname{grad} F(u)$.

If

$$F(u + h) - F(u) - (f(u), h) \geq \frac{1}{2}(b - c)\|h\|^2$$

or

$$(f(u + h) - f(u), h) \geq (b - c)\|h\|^2,$$

where $b\lambda_1 > 1$, $c\|T\| < 1$, and $T = (bK - I)^{-1}K$, then there exists one and only one solution of the equation

$$u = Kf(u).$$

2. Nonlocal implicit function theorem.

Theorem 2. Let the functions $g_i(t, v_1, \dots, v_n, u_1, \dots, u_m)$ $(i = 1, \dots, m)$ satisfy the Carathéodory condition.

If

$$\sum_{i=1}^{m} [g_i(t, v_1, \dots, v_n, u_1 + h_1, \dots, u_m + h_m)$$

$$-g_i(t, v_1, \dots, v_n, u_1, \dots, u_m)]h_i \geq \varepsilon_0 \sum_{i=1}^{m} h_i^2 \qquad (\varepsilon_0 > 0), \tag{4}$$

then there exists one and only one system of functions

$$u_i = \eta_i(t, v_1, \dots, v_n) \qquad (i = 1, \dots, m), \tag{5}$$

satisfying the equations

$$g_i(t, v_1, \dots, v_n, u_1, \dots, u_m) = 0 \qquad (i = 1, \dots, m),$$

and each of the functions (5) satisfies the Carathéodory condition.

3. The general case.

In this section we consider only the case of Hilbert space.

Let K_i be bounded self-adjoint operators acting in L_2. Let $(-a_i, a_i + \lambda_i)$ be a subinterval of the resolvent set of the operator K_i. Put $\widetilde{K}_i = K_i + a_i I$, $T_i = (b_i \widetilde{K}_i - I)^{-1} \widetilde{K}_i$, where $b_i \lambda_i > 1$. Then the T_i are bounded positive definite operators. Let $c_i > 0$ satisfy the inequalities $c_i \|T\| < 1$.

Theorem 3. Let K_i be the operators defined above, let the functions $f_i(t, u_1, \dots, u_m)$ satisfy conditions δ_1) and δ_2) of item 1, and let the operator $f(u)$ be continuous in L_2.

Let:

$$\sum_{i=1}^{m} a_i [f_i(t, u_1 + h_1, \dots, u_m + h_m) - f_i(t, u_1, \dots, u_m)]h_i \geq (\varepsilon_0 - 1) \sum_{i=1}^{m} h_i^2, \tag{I}$$

where ε_0 is a sufficiently small positive number,

$$F(t, u_1 + h_1, \dots, u_m + h_m) - F(t, u_1, \dots, u_m) -$$

$$- \sum_{i=1}^{m} \{1 - a_i(b_i - c_i)\} [f_i(t, u_1 + h_1, \dots, u_m + h_m) - f_i(t, u_1, \dots, u_m)]h_i$$

$$\geq \frac{1}{2} \sum_{i=1}^{m} (b_i - c_i) h_i^2 + \sum_{i=1}^{m} a_i [f_i(t, u_1 + h_1, \dots, u_m + h_m) - f_i(t, u_1, \dots, u_m)]^2. \tag{II}$$

Then there exists one and only one solution of system (2).

Remark 1. The assertion of the theorem remains valid if condition (II) is replaced by the condition

$$\sum_{i=1}^{m}[1-2a_i(b_i-c_i)]\,[f_i(t,u_1+h_1,\dots,u_m+h_m)-f_i(t,u_1,\dots,u_m)]\,h_i$$

$$\geq \sum_{i=1}^{m}(b_i-c_i)h_i^2+\sum_{i=1}^{m}[a_i^2(b_i-c_i)-a_i]\,[f_i(t,u_1+h_1,\dots,u_m+h_m)-$$

$$-f_i(t,u_1,\dots,u_m)]^2. \tag{II'}$$

Remark 2. The simultaneous fulfillment of conditions (I) and (II), or (I) and (II′), is possible. For example, suppose that $1-2a_i(b_i-c_i)>0$, $b_i\geq c_i$. Then (I) and (II′) are simultaneously fulfilled if

$$\sum_{i=1}^{m}[f_i(t,u_1+h_1,\dots,u_m+h_m)-f_i(t,u_1,\dots,u_m)]\,h_i\geq$$

$$\geq\sum_{i=1}^{m}\frac{b_i-c_i}{\alpha}h_i^2,$$

where $\alpha=\min_i[1-2a_i(b_i-c_i)]$.

For the proof of Theorem 3, put

$$g_i(t,u_1,\dots,u_m)=u_i+a_if_i(t,u_1,\dots,u_m).$$

Then system (2) is equivalent to the vector equation

$$g(u)=\widetilde{K}f(u), \tag{6}$$

where $\widetilde{K}=(\widetilde{K}_1,\dots,\widetilde{K}_m)$.

Since

$$\sum_{i=1}^{m}[g_i(t,u_1+h_1,\dots,u_m+h_m)-g(t,u_1,\dots,u_m)]\,h_i\geq\varepsilon_0\sum_{i=1}^{m}h_i^2, \tag{7}$$

it follows, by Theorem 2, that there exists one and only one system of functions

$$u_i=\eta_i(t,v_1,\dots,v_m)\qquad(i=1,2,\dots,m),$$

satisfying the equations

$$v_i = g_i(t, u_1, \dots, u_m).$$

It follows from (7) that the functions $\eta_i(t, v_1, \dots, v_m)$ satisfy the Lipschitz condition. Therefore the operator $\eta(v)$ is continuous in L_2.

Put

$$\tilde{f}_i(t, v_1, \dots, v_m) = f_i(t, \eta_1(t, v_1, \dots, v_m), \dots, \eta_m(t, v_1, \dots, v_m))$$

$$(i = 1, \dots, m),$$

$$\tilde{f}(v)(t) = (\tilde{f}_1(t, v_1(t), \dots, v_m(t)), \dots, \tilde{f}_m(t, v_1(t), \dots, v_m(t))).$$

Then the operator $\tilde{f}(v)$ is continuous in L_2, and equation (6) is equivalent to the equation

$$v = \widetilde{K}\tilde{f}(v).$$

Theorem 1′ is applicable to this equation.

The potentiality of the operator $\tilde{f}(v)$ follows from the following assertion.

Lemma. There exists a function $\widetilde{F}(t, v_1, \dots, v_m)$, satisfying the Carathéodory condition, such that $\widetilde{F}(t, 0, \dots, 0) \in L$ and

$$\frac{\partial}{\partial v_i}\widetilde{F}(t, v_1, \dots, v_m) = \tilde{f}_i(t, v_1, \dots, v_m).$$

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Note: Figure translations are in progress. See original paper for figures.

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