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Abstract

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THE FIRST BOUNDARY-VALUE PROBLEM FOR DEGENERATE QUASILINEAR ELLIPTIC SYSTEMS OF DIFFERENTIAL EQUATIONS

(Presented by Academician S. L. Sobolev, 23 I 1964)

In this paper the solvability of the first boundary-value problem is proved for a certain class of nonlinear elliptic systems of order $2m$ admitting degeneracy. The distinctive feature of the systems considered is that their generalized solutions may fail to have derivatives of order m that are square-summable; however, derivatives of certain powers of derivatives of order $m-1$ exist and belong to L_2. We also note that, in contrast to "fixed"degeneracy ([1,2]), no loss of boundary conditions occurs. The proof is carried out by a method analogous to Galerkin' s method, which has been developed in detail in a number of works by M. I. Vishik ([3–6]). In this connection one simple embedding theorem is used.

§ 1. Some integral inequalities

Let G be a bounded domain of the n-dimensional Euclidean space E_n, satisfying the cone condition ([7]); let Γ be the boundary of G; $C^m(\overline{G})$ the space of functions defined in $\overline{G}$ and having bounded derivatives up to order m. Notation:

$$[u]_G = \int_G u(x)\,dx; \qquad [u]_\Gamma = \int_\Gamma u(x')\,d\gamma \quad (x' \in \Gamma).$$

Lemma 1. Let $-\infty < \alpha_0 < +\infty,\ \alpha_1 \geqslant 1;\quad u(x),\ |u(x)|^{\alpha_0+\alpha_1} \in C^1(\overline{G})$. Then the inequality* holds

$$\left[|u|^{\alpha_0+\alpha_1}\right]_G \leqslant K\left(\left[|u|^{\alpha_0}\left|\frac{\partial u}{\partial x_i}\right|^{\alpha_1}\right]_G + \left[|u|^{\alpha_0+\alpha_1}\right]_\Gamma\right) \quad (i = 1, \dots, n). \quad (1)$$

The constant K depends on α_0, α_1, and the domain G.

Proof. Integrating the obvious equality

$$-x_i \frac{\partial}{\partial x_i} |u|^{\alpha_0+\alpha_1} = -(\alpha_0+\alpha_1) x_i |u|^{\alpha_0+\alpha_1-1} \operatorname{sgn} u \frac{\partial u}{\partial x_i},$$

we obtain

$$\left[|u|^{\alpha_0+\alpha_1}\right]_G \leqslant K \left(\left[|u|^{\alpha_0+\alpha_1-1} \left| \frac{\partial u}{\partial x_i} \right| \right]_G + \left[|u|^{\alpha_0+\alpha_1}\right]_\Gamma \right).$$

To prove (1), it remains only to use Young' s inequality $ab \leqslant p^{-1}\varepsilon^p a^p + q^{-1}\varepsilon^{-q} b^q$ $(a, b \geqslant 0,\ p^{-1} + q^{-1} = 1)$, with $q = \alpha_1$ and a suitable ε.

Lemma 2. Let $-\infty < \alpha_0 < +\infty$, $\alpha_1 \geqslant 0$, $\alpha_2 \geqslant 0$, $\alpha_1 + \alpha_2 \geqslant 1$; $u(x)$, $|u(x)|^{\alpha_0+\alpha_1+\alpha_2} \in C^1(\overline{G})$. Then the inequality holds

$$\left[|u|^{\alpha_0+\alpha_1} \left| \frac{\partial u}{\partial x_i} \right|^{\alpha_2} \right]_G \leqslant K \left(\left[|u|^{\alpha_0} \left| \frac{\partial u}{\partial x_i} \right|^{\alpha_1+\alpha_2} \right]_G + \left[|u|^{\alpha_0+\alpha_1+\alpha_2}\right]_\Gamma \right) \quad (i = 1, \dots, n). \tag{2}$$

Proof of Lemma 2 follows from Young' s inequality with exponent $q = (\alpha_1 + \alpha_2)/\alpha_2$ and Lemma 1.

Lemma 3. Let $u(x) \in C^1(\overline{G})$, $\alpha_0 \geqslant 0$, $\alpha_1 \geqslant 1$. Then:

a) If $\alpha_1 < n$, then

$$\left[|u|^{(\alpha_0+\alpha_1)\frac{n}{n-\alpha_1}} \right]_G^{\frac{n-\alpha_1}{n}} \leqslant K \left(\sum_{i=1}^n \left[|u|^{\alpha_0} \left| \frac{\partial u}{\partial x_i} \right|^{\alpha_1} \right]_G + \left[|u|^{\alpha_0+\alpha_1}\right]_\Gamma \right);$$

* For $\alpha_0 = 0$, $\alpha_1 = 2$, inequality (1) is known as Friedrichs' inequality. In the case where α_0 and α_1 are even and $u|_\Gamma = 0$, inequality (1) was obtained earlier in ([4]) by M. I. Vishik.

b) if $\alpha_1 = n$, then

$$\left[|u|^{(\alpha_0+\alpha_1)p} \right]_G^{1/p} \leq K \left(\sum_{i=1}^n \left[|u|^{\alpha_0} \left| \frac{\partial u}{\partial x_i} \right|^{\alpha_1} \right]_G + \left[|u|^{\alpha_0+\alpha_1}\right]_\Gamma \right),$$

where $p \geq 1$ is arbitrary *;

c) if $\alpha_1 > n$, then

$$\max_{x\in G}|u| \le K\left(\sum_{i=1}^{n}\left[|u|^{\alpha_0}\left|\frac{\partial u}{\partial x_i}\right|^{\alpha_1}\right]_G + [|u|^{\alpha_0+\alpha_1}]_\Gamma\right)^{1/(\alpha_0+\alpha_1)}.$$

In this case, from the uniform boundedness of the right-hand sides there follows compactness of the family $u(x)$, respectively, in the spaces L_q $(q < (\alpha_0 + \alpha_1)n/(n-\alpha_1))$, L_p, and $C(G)$.

Proof. Consider the function $v(u) = |u|^{1+\alpha_0/\alpha_1}\operatorname{sgn} u$. From Lemma 1 we obtain that $\|v\|_{W^{(1)}_{\alpha_1}}$ is estimated by the right-hand side of inequality a). After this the inequalities of Lemma 3 follow from the embedding theorems of S. L. Sobolev ([7]).

Let us now prove compactness, for example, in case a). From the same embedding theorems it follows that there exists a sequence $v(u_n)$ which converges in L_r $(r < n\alpha_1/(n-\alpha_1))$ to some function $v(u)$. As is known, from a sequence converging in the mean one can select a subsequence converging almost everywhere. Since $v(u)$ depends monotonically on u, there exists a sequence $u_m(x)$ converging almost everywhere to some function $u(x)$. It is easy to see that $u(x) \in L_q$. From the Vallée-Poussin theorem ([8]) and inequality a) it follows that $[\,|u_m|^q\,]_G$ are uniformly absolutely continuous; therefore, from convergence almost everywhere there follows convergence in the L_q norm. Lemma 3 in case a) is completely proved.

Compactness in cases b) and c) is proved just as easily.

Remark. If $\alpha_i \ge 0$, $\beta_i \ge 1$,

$$p = n - 1 - \sum_{i=1}^{n}(\beta_i - 1)/\beta_i,$$

$p > 0$, $u \in L_q(E_n) \cap C^1(E_n)$ $(q > 0)$, then the inequality

$$\left[|u|^{\frac{1}{p}\sum_{i=1}^{n}(1+\alpha_i/\beta_i)p}\right]_{E_n} \le K\prod_{i=1}^{n}\left[|u|^{\alpha_i}\left|\frac{\partial u}{\partial x_i}\right|^{\beta_i}\right]_{E_n}^{1/\beta_i}$$

is valid.

§ 2. Systems of differential equations.

Consider the system of equations

$$\mathcal{L}(u) \equiv \sum_{|\alpha'|,|\alpha|\le m}(-1)^{|\alpha'|}D^{\alpha'}\left(A_\alpha^{\alpha'}(x, D^\gamma u)D^\alpha u\right) + \sum_{|\beta|=m}V_\beta(x, D^\gamma u)D^\beta u +$$

$$+\sum_{|\delta|\le m}(-1)^{|\delta|}D^{\delta}V_{\delta}(x,D^{\gamma}u)=0\qquad(|\gamma|\le m-1);\tag{3}$$

$$D^{\omega}u\big|_{\Gamma}=f_{\omega}(x'),\qquad x'\in\Gamma,\qquad |\omega|\le m-1.\tag{4}$$

Here $\alpha=(\alpha_1,\dots,\alpha_n)$ is a multi-index of differentiation; $D^{\alpha}=D_1^{\alpha_1}\cdots D_n^{\alpha_n}$, $D_i=\partial/\partial x_i$, $|\alpha|=\alpha_1+\cdots+\alpha_n$; $D^0\equiv E$ (the identity operator). The equality $\mu=\alpha-1$ will mean that all possible derived expressions of the form

$$D^{\mu}=D_1^{\alpha_1}\cdots D_i^{\alpha_i-1}\cdots D_n^{\alpha_n},\qquad |\mu|=|\alpha|-1,$$

are taken. Similarly for α', β, γ, and δ. Further, $u(x)=(u_1,\dots,u_N)$, $A_{\alpha}^{\alpha'}(x,D^{\gamma}u)$ and $V_{\beta}(x,D^{\gamma}u)$ are square matrices of order N; $V_{\delta}(x,\xi_{\gamma})=(V_{\delta}^1,\dots,V_{\delta}^N)$, $\xi_{\gamma}=(\xi_{\gamma}^1,\dots,\xi_{\gamma}^N)$. Finally, $f_{\omega}(x')=(f_{\omega}^1,\dots,f_{\omega}^N)$. Thus, (3), (4) is the first boundary–

* This assertion can be strengthened in terms of Orlicz spaces (see ([9])).

boundary-value problem for a system of N equations with N unknown functions $u_1(x),\dots,u_N(x)$.

Assumptions.

I. Ellipticity condition. For any smooth function $u(x)$ the inequality

$$\mathcal{L}(u,u)=\sum_{|\alpha'|,|\alpha|\le m}[A_{\alpha}^{\alpha'}(x,D^{\gamma}u)D^{\alpha}u,D^{\alpha'}u]_G+\sum_{|\beta|=m}[V_{\beta}(x,D^{\gamma}u)D^{\beta}u,u]_G$$
$$+\sum_{|\delta|\le m}[V_{\delta}(x,D^{\gamma}u),D^{\delta}u]_G\ge a_0\sum_{|\alpha|=m,\ \mu=\alpha-1}[|D^{\mu}u|^{p_{\mu}}|D^{\alpha}u|^2]_G-K\equiv E(u)-K,$$

holds, where $a_0>0,\ p_{\mu}\ge 0$ are certain numbers.

II. Conditions on the "coefficients." $A_{\alpha}^{\alpha'}(x,D^{\gamma}u),V_{\beta}(\dots),V_{\delta}(\dots)$ are continuous functions of $D^{\gamma}u$, and:

1. $$|A_{\alpha}^{\alpha'}(x,D^{\gamma}u)|\le K_1\sum_{\mu=\alpha-1}l_{\alpha}^{\mu}(x,D^{\gamma}u)|D^{\mu}u|^{p_{\mu}},$$

   where $l_{\alpha}^{\mu}(x,\xi_{\gamma})$ are arbitrary functions continuous in ξ_{γ}, satisfying the inequality

   $$|l_{\alpha}^{\mu}(x,\xi_{\gamma})|\le K_2\sum_{|\gamma|\le m-1}|\xi_{\gamma}|^{p_{\alpha,\mu}}+K_3,\qquad 0\le p_{\alpha,\mu}<1.$$

2. $$|V_{\beta}(x,D^{\gamma}u)|\le a_{\beta}(x)\prod_{|\omega|=0}^{m-1}|D^{\omega}u|^{i_{\omega}},\qquad i_{\omega}\ge 0,\quad \sum_{|\omega|=0}^{m-1}i_{\omega}<p_{\beta}+1,$$

and, if $|\omega| = m-1$, then $\omega = \beta - 1$ and $2i_\omega \geq p_\beta$.

3.

$$|V_\delta(x, D^\gamma u)| \leq b_\delta(x) \prod_{|\omega|=0}^{m-1} |D^\omega u|^{i_\omega}, \qquad i_\omega \geq 0, \quad \sum_{|\omega|=0}^{m-1} i_\omega < p_\delta + 2.$$

The functions $a_\beta(x)$ and $b_\delta(x)$ are summable to some power depending on p_μ and n (we do not write it out). We note that the right-hand sides of inequalities (2) and (3) may be replaced by a finite number of terms of this type.

III. There exists a function $f(x)$ ($x \in \overline{G}$), bounded together with all derivatives up to order m, such that $D^\omega f|_\Gamma = f_\omega(x')$.*

Definition. A function $u(x)$ is called a generalized solution of problem (3), (4) if:

1. $D_i(|D^\mu u|^{1+p_\mu/2} \operatorname{sgn} D^\mu u) \in L_2 \quad (i = 1, \dots, n)$.
2. $D^\omega(u-f)|_\Gamma = 0$ in the mean (see (7)).
3. For any function $v(x) \in \overset{0}{C}{}^m(\overline{G})$ the equality

$$\mathcal{L}(u, v) \equiv \sum_{|\alpha'|,|\alpha| \leq m} [A_\alpha^{\alpha'}(x, D^\gamma u) D^\alpha u, D^{\alpha'} v]_G + \sum_{|\beta|=m} [V_\beta(x, D^\gamma u) D^\beta u, v]_G + \sum_{|\delta| \leq m} [V_\delta(x, D^\gamma u), D^\delta v]_G = 0 \tag{5}$$

holds.

Theorem. *If conditions I–III are fulfilled, then problem (3), (4) has at least one generalized solution.*

We outline the proof of the theorem. Let $\{v_k(x)\}$ be a system of smooth vector functions complete in $\overset{0}{C}{}^m(G)$. We seek an approximate solution of problem (3), (4) in the form

$$u_m(x) = f(x) + z_m(x), \qquad z_m(x) = \sum_{k=1}^m c_{km} v_k(x).$$

The unknown numbers c_{km} are determined from the system of nonlinear algebraic equations

$$[\mathcal{L}(f + z_m), v_k]_G = 0 \qquad (k = 1, \dots, m). \tag{6}$$

* We omit the case when $f_\omega(x')$ admits an extension in G, summable to some power depending on p_μ. It is treated analogously.

The solvability of this system follows from the lemma of paper ([4]). In doing so, the following energy estimate is used:

$$[\mathcal{L}(f + z_m), z_m]_G \equiv \mathcal{L}(u_m, z_m) \geq E(u_m) - K,$$

where K depends on $f(x)$. For the proof one must multiply (6) by c_{km}, sum over k from 1 to m, integrate by parts, and take into account conditions I-III. Here Lemmas 2, 3 and the embedding theorems are used essentially. From the estimate obtained, Lemma 3 and the embedding theorems it follows that there exists a subsequence $u_r(x)$ and a function $u(x)$ such that

$$u_r(x) \to u(x), \dots, D^\mu u_r \to D^\mu u$$

almost everywhere in G, with $D^\mu u \in L_q(q \le (p_\mu + 2)n/(n-2))$. By virtue of the weak compactness of the sphere in L_2,

$$D_i(|D^\mu u|^{1+p_\mu/2} \operatorname{sgn} D^\mu u) \in L_2$$

and

$$D_i(|D^\mu u_r|^{1+p_\mu/2} \operatorname{sgn} D^\mu u_r) \to D_i(|D^\mu u|^{1+p_\mu/2} \operatorname{sgn} D^\mu u)$$

weakly in L_2. Moreover, $D^\omega(u-f)|_\Gamma = 0$ in the mean. The function $u(x)$ found is a solution of problem (3), (4), i.e. it satisfies identity (5). To verify this, it suffices to show that in (6) (after integration by parts) passage to the limit under the integral sign is possible. This follows from conditions I, II and the following lemma.

Lemma 4. Let, as $r \to \infty$, $u_r \to u$ almost everywhere in G, and

$$|u_r|^{p/2} D_i u_r \to \frac{2}{2+p} D_i(|u|^{1+p/2} \operatorname{sgn} u)$$

weakly in L_2. Then, if $m(x,u)$ is a function continuous in u and $\|m(x,u_r)\|_q \le K$ for some $q > 2$, then

$$m(x,u_r)|u_r|^{p/2} D_i u_r \to \frac{2}{2+p} m(x,u) D_i(|u|^{1+p/2} \operatorname{sgn} u)$$

weakly in L_1.

Thus the solvability of problem (3), (4) is proved.

Example $(N = 1)$.

$$\mathcal{L}(u) \equiv \sum_{i,k=1}^{n} \frac{\partial^2}{\partial x_i \partial x_k} \left(\sum_{l=1}^{n} \delta_{ikl} \left| \frac{\partial u}{\partial x_l} \right|^p \frac{\partial^2 u}{\partial x_i \partial x_k} \right) + \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} \left(a_i(x) |u|^{\alpha_i} \left| \frac{\partial u}{\partial x_i} \right|^{\beta_i} \right)$$
$$+ \sum_{i=1}^{n} b_i(x) |u|^{\gamma_i} \left| \frac{\partial u}{\partial x_i} \right|^{\delta_i} \frac{\partial^2 u}{\partial x_i^2} + h(x),$$

where $\delta_{ikl} = \{0, \text{ if } l \ne k,\ l \ne i;\ 1, \text{ if } l = k \text{ or } l = i\}$; $\alpha_i + \beta_i < p + 1$, $\gamma_i + \delta_i < p$, $a_i(x) \in L_{r_i}$, where r_i are determined by the relations

$$r_i^{-1} + p_1^{-1} + p_2^{-1} + p_3^{-1} = 1, \qquad p_1 = 2, \qquad p_2 = \frac{2n(p+2)}{(n-2)(2\beta_i - p)},$$

$$p_3 = \frac{n(p+2)}{(n-p-4)\alpha_i};$$

$b_i(x) \in L_{m_i}$, where m_i are determined as are r_i, with α_i replaced by $\gamma_i + 1$ and β_i by δ_i; $h(x) \in L_{\frac{p+2}{p+1}}$.

With some changes, mainly concerning Lemma 3, the method indicated is applicable to operators of the type

$$\mathcal{L}(u) \equiv -\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(|u|^{\alpha_i}\frac{\partial u}{\partial x_i}\right) + \sum_{i=1}^{n} V_i(x,u)\frac{\partial u}{\partial x_i} + h(x), \qquad \alpha_i \geq 0.$$

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Note: Figure translations are in progress. See original paper for figures.