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Abstract

Full Text

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ON MULTIDIMENSIONAL LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

(Presented by Academician I. G. Petrovskii, 15 X 1963)

MATHEMATICS

1. Let E_x be an m -dimensional space over the field P_x , and let E_y be an n -dimensional space over the field P_y . Suppose that $P_{xy} = P_x \cap P_y$ is also a field. An operator a acting from E_x into E_y will be called **linear** if $a(x_1 + x_2) = ax_1 + ax_2$ ($x_1, x_2 \in E_x$) and $a(\alpha x) = \alpha ax$ for $\alpha \in P_{xy}$. The set of linear operators from E_x into E_y , under the usual definitions of the operations of addition and multiplication by scalars from P_y , forms a linear space over the field P_y , which we shall denote by E_{xy} . In what follows we shall encounter the spaces $E_{y(xy)}$ and $E_{x(yy)}$. The fields that occur will be either the field of complex numbers or the field of real numbers.

Let $A \in E_{y(xy)}$. The operator $\tilde{A} \in E_{x(yy)}$, uniquely determined by the equation $(Ay)x = (\tilde{A}x)y$ ($x \in E_x$, $y \in E_y$), will be called the **adjoint**. The adjoint operator to an operator $B \in E_{x(yy)}$ is defined analogously. We note that in the case when the spaces E_x and E_y are endowed with norms, the equality $\|A\| = \|\tilde{A}\|$ holds.

Let $E'_y = E_\varphi$ be the space conjugate to E_y . An operator $A' \in E_{\varphi(xy)}$ will be called **conjugate** to the operator $A \in E_{y(xy)}$ if $((Ay)x, \varphi) = (y, (A'\varphi)x)$ for all $x \in E_x$, $y \in E_y$, and $\varphi \in E_\varphi$.

2. Let E_x be an m -dimensional space and E_y an n -dimensional space (each of them may be either real or complex). Consider the Cauchy problem for the linear equation ($A \in E_{y(xy)}$)

$$\frac{dy}{dx} = Ay, \quad (1)$$

$$y(\xi) = \eta \quad (\xi \in E_x, \eta \in E_y). \quad (2)$$

Theorem 1. *In order that the Cauchy problem (1)–(2) be solvable for every η , it is necessary and sufficient that the condition*

$$\tilde{A}h \tilde{A}k = \tilde{A}k \tilde{A}h, \quad h, k \in E_x \quad (3)$$

be satisfied.

If condition (3) is satisfied, the solution is written in the form

$$y(x) = e^{\tilde{A}(x-\xi)}\eta \quad (4)$$

and can be obtained by the usual method of successive approximations:

$$y_0(x) \equiv \eta; \quad \frac{dy_{k+1}}{dx} = Ay_k, \quad y_{k+1}(\xi) = \eta \quad (k = 0, 1, 2, \dots). \quad (5)$$

We note that condition (3) is superfluous if $m = 1$. In the case when E_x and E_y are real, Theorem 1 follows from the general theorems of the work ⁽¹⁾.

Alongside problem (1)–(2), it is useful simultaneously to study the Cauchy problem for the **conjugate** equation

$$\frac{d\varphi}{dx} = -A'\varphi, \quad (6)$$

$$\varphi(\xi) = \theta \quad (\xi \in E_x, \theta \in E_\varphi). \quad (7)$$

Theorem 2. If problem (1)–(2) is solvable for arbitrary η , then problem (6)–(7) is solvable for arbitrary θ , and conversely,—if problem (6)–(7) is solvable for arbitrary θ , then problem (1)–(2) is solvable for arbitrary η .

When the solvability condition is fulfilled, the solution of problem (6)–(7) is written in the form

$$\varphi(x) = e^{-\tilde{A}'(x-\xi)}\theta = e^{-[\tilde{A}(x-\xi)]'\theta} \quad (8)$$

and can be obtained by the method of successive approximations:

$$\varphi_0(x) \equiv \theta; \quad \frac{d\varphi_{k+1}}{dx} = -A'\varphi_k, \quad \varphi_{k+1}(\xi) = \theta \quad (k = 0, 1, 2, \dots). \quad (9)$$

The characteristic property of adjoint equations $(y(x), \varphi(x)) \equiv \text{const}$ also holds in our case.

3. Let us now consider the Cauchy problem for operator equations

$$\frac{dY}{dx} = \tilde{A}Y, \quad Y(\xi) = H, \quad (10)$$

$$\frac{d\Phi}{dx} = -\tilde{A}'\Phi, \quad \Phi(\xi) = \theta. \quad (11)$$

It is easy to show that from the solvability of problem (1)–(2) for arbitrary η there follows the solvability of problem (10) for arbitrary H , and conversely. Further, from the solvability of problem (10) for arbitrary H there follows the solvability of problem (11) for arbitrary θ ; the converse is true. The solutions of problems (10) and (11) can be represented in the form

$$Y(x) = e^{\tilde{A}(x-\xi)}H, \quad \Phi(x) = e^{-[\tilde{A}(x-\xi)]'\theta}. \quad (12)$$

The operator function $Y(x; \xi) = e^{\tilde{A}(x-\xi)}$ will be called the **fundamental** operator-function (of equation (1)).

The solution $Y(x) = e^{\tilde{A}x}$ of problem (10) for $\xi = 0$, $H = 1$, as is easy to see, satisfies the following conditions

- 1) $Y(0) = 1$;
- 2) $Y(x)$ is continuous at least for one value of x ;
- 3) $Y(x + \xi) = Y(x)Y(\xi)$ for $x, \xi \in E_x$.

For $m = 1$, Pólya's theorem⁽²⁾ is known, according to which an operator function satisfying conditions 1)–3) can be written in the form $Y(x) = e^{x^a}$. A generalization of this theorem to the case $m > 1$ is

Theorem 3. Let an operator function $Y(x)$, defined for all $x \in E_x$, satisfy conditions 1)–3). Then it can be represented in the form

$$Y(x) = e^{\tilde{A}x}, \quad (13)$$

where

$$\tilde{A}x = \lim_{\varepsilon \rightarrow 0} \frac{Y(\varepsilon x) - 1}{\varepsilon} \quad (14)$$

and \tilde{A} satisfies condition (3).

This theorem is proved according to the scheme proposed in (3).

4. In all that follows we shall assume the space E_x to be real, and the space E_y to be complex.

Let $A \in E_{y(xy)}$. A vector $f \in E_y$ is called an **eigenvector** of the operator A (\tilde{A}), if one can indicate a linear functional $\lambda(x)$ for which, for all $x \in E_x$, the equality holds

$$(Af)x = \lambda(x)f \quad \text{or} \quad (\tilde{A}x)f = \lambda(x)f. \quad (15)$$

The complex-valued functional $\lambda(x)$ is called an eigenfunctional of the operator A (\tilde{A}).

Theorem 4. Let the operator \tilde{A} satisfy condition (3). Then one can specify subspaces E_1, \dots, E_s such that

$$E_y = E_1 \oplus \dots \oplus E_s. \quad (16)$$

Each of the subspaces E_j is invariant with respect to the operators $\tilde{A}x$ ($j = 1, \dots, s$); denote the induced operator by \tilde{A}_jx ($j = 1, \dots, s$). Then

$$\tilde{A}x = \tilde{A}_1x \oplus \dots \oplus \tilde{A}_{sx}. \quad (17)$$

Each of the operators \tilde{A}_jx is represented in the form

$$\tilde{A}_jx = \lambda_j(x)\mathbf{1}_j + \tilde{T}_jx, \quad (18)$$

where all the eigenfunctionals $\lambda_j(x)$ ($j = 1, \dots, s$) are pairwise distinct, $\mathbf{1}_j$ is the identity operator in the space E_j ($j = 1, \dots, s$), and \tilde{T}_jx is a nilpotent-valued operator ($j = 1, \dots, s$).

It follows from this theorem that $e^{\tilde{A}x}$ can be represented in the form

$$e^{\tilde{A}x} = e^{\tilde{A}_1x} \oplus \dots \oplus e^{\tilde{A}_{sx}}, \quad (19)$$

$$e^{\tilde{A}_jx} = e^{\lambda_j(x)} \left\{ \mathbf{1}_j + \tilde{T}_jx + \dots + \frac{(\tilde{T}_jx)^{n_j-1}}{(n_j-1)!} \right\}, \quad (20)$$

where n_j is the dimension of the space E_j ($j = 1, \dots, s$).

Our subsequent results are connected with the decompositions (19) and (20).

5. In this item we shall determine when all solutions of the operator equation

$$\frac{dY}{dx} = \tilde{A}Y \quad (21)$$

are bounded, almost periodic, or periodic. Here an operator function is called periodic if the linear span of its group of periods coincides with the entire space on which it is defined. (For the definition of almost periodic functions defined on a group, see the monograph ⁽⁴⁾.)

From formulas (19) and (20) it follows that $\exp \tilde{A}x$ is bounded if and only if the equalities

$$\operatorname{Re} \lambda_j(x) = 0, \quad \tilde{T}_j x = 0, \quad j = 1, \dots, s; \quad x \in E_x \quad (22)$$

hold.

When these restrictions are satisfied, the decomposition (19) takes the form

$$\exp \tilde{A}x = \exp(i \operatorname{Im} \lambda_1(x)) \mathbf{1}_1 \oplus \dots \oplus \exp(i \operatorname{Im} \lambda_s(x)) \mathbf{1}_s, \quad (23)$$

from which it follows immediately that a bounded solution of equation (21) is always almost periodic.

It is of interest to find conditions under which the operator function $\exp \tilde{A}x$ will be periodic. Let $\nu_j(x)$ ($j = 1, \dots, s$) be the imaginary parts of the eigenfunctionals $\lambda_1, \dots, \lambda_s$. Let L be the linear span of the functionals ν_1, \dots, ν_s in the conjugate space E'_x . It is easy to show that the annihilator L^0 is a maximal constant manifold of the function $\exp \tilde{A}x$ ⁽⁵⁾ (for the terminology and notation of the present article, see ⁽⁶⁾).

Theorem 5. The operator function $\exp \tilde{A}x$ is periodic if and only if in L one can specify a basis e_1, \dots, e_k for which

$$v_j = r_j^1 e_1 + \dots + r_j^k e_k \quad (j = 1, \dots, s), \quad (24)$$

where the coefficients r_j^i are rational numbers.

6. Suppose that the space E_y is normed. A direction $h \in E_x$ will be called a **direction of boundedness** if, with c_h a constant, $\|\exp(t\tilde{A}h)\| \leq c_h$ for $0 \leq t < +\infty$. The totality of such h forms a convex pointed cone—the **cone of boundedness**.

A direction $h \in E_x$ will be called a **direction of tending to zero** if $\|\exp(t\tilde{A}h)\| \rightarrow 0$ as $t \rightarrow +\infty$. The totality of such h forms a convex blunt cone—the **cone of tending to zero**. It is not difficult to see that the cone is determined by the inequalities

$$\operatorname{Re} \lambda_j(x) < 0, \quad j = 1, \dots, s, \quad (25)$$

where $\{\lambda_j\}$ are all the eigenfunctionals of the operator A . According to Carver's theorem (⁷), the system of inequalities (25) is consistent if and only if the zero functional is not contained in the convex hull of the functionals $\operatorname{Re} \lambda_j(x)$ ($j = 1, \dots, s$).

In conclusion we formulate one more theorem.

Theorem 6. The following equality holds:

$$\lim_{t \rightarrow +\infty} \frac{\ln \|\exp(t\tilde{A}x)\|}{t} = \max_{1 \leq j \leq s} \operatorname{Re} \lambda_j(x). \quad (26)$$

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Note: Figure translations are in progress. See original paper for figures.

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