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Abstract

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ESTIMATES OF THE NUMBER OF STATES ARISING IN THE DETERMINIZATION OF A NONDETERMINISTIC AUTONOMOUS AUTOMATON

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Let $\mathfrak{A}$ be a nondeterministic autonomous automaton with states $\sigma_1, \sigma_2, \dots, \sigma_n$, i.e., a multivalued mapping of the set S of states into itself*:

$$\sigma_i \to S_i; \qquad S_i \subset S \qquad (i = 1, 2, \dots, n). \tag{1}$$

We fix one of the states, for example σ_1, as the initial state.

The mapping (1) is directly extended to the system Σ of all subsets of the set S:

$$M \to M' = \bigcup_{\sigma_i \in M} S_i \qquad (M \in \Sigma). \tag{2}$$

Restricting the mapping (2) to the smallest subsystem $\Sigma_0 \subset \Sigma$ invariant with respect to it and containing the "initial" set $\{\sigma_1\}$, we obtain a certain already deterministic autonomous automaton $\mathfrak{A}_0$. The passage from $\mathfrak{A}$ to $\mathfrak{A}_0$ is naturally called the determinization of the (initial) automaton $\mathfrak{A}$.

The determinization procedure is used, for example, in ([1]) in constructing an automaton representing a given regular event**. In this and in certain other related problems there arises the question of estimating the number of states of the automaton $\mathfrak{A}_0$. We shall denote this number by $d(\mathfrak{A})$. We shall obtain for $d(\mathfrak{A})$ an estimate substantially better than the trivial estimate $d(\mathfrak{A}) \leqslant 2^n$. Moreover, we shall show how $d(\mathfrak{A})$ can be estimated while taking into account the structure of the automaton $\mathfrak{A}$.

Consider the Boolean algebra of square matrices of order n with entries $0, 1$, and associate with the automaton $\mathfrak{A}$ the matrix*** $A = (a_{ik})_{i,k=1}^n$, putting $a_{ik} = 1$ if $\sigma_i \in S_k$, and $a_{ik} = 0$ otherwise. Next consider the sequence of powers

$$A^m = (a_{ik}^{(m)})_{i,k=1}^n \qquad (m = 0, 1, 2, ...). \tag{3}$$

It is easy to see that the states of the automaton $\mathfrak{A}_0$ are all possible sets of the form

$$\bigcup_{i=1}^{n} a_{i1}^{(m)} \{\sigma_i\} \qquad (m = 0, 1, 2, ...).$$

Here we put $\varepsilon M = M$ for $\varepsilon = 1$ and $\varepsilon M = \emptyset$ for $\varepsilon = 0$ for all $M \in \Sigma$. It is now clear that

$$d(\mathfrak{A}) \leqslant l(A), \tag{4}$$

where $l(A) = t(A) + p(A)$, $p(A)$ is the length of the least period of the sequence (3), and $t(A)$ is the length of its preperiod.

* That is, a finite graph.

** Our case of an autonomous automaton corresponds to the representation of events over a one-letter alphabet.

*** The adjacency matrix in the terminology of graph theory.

To estimate the value $l(A)$ we use the theory of matrices with nonnegative entries ([2–6]), a number of whose concepts and theorems carry over directly to the Boolean matrices under consideration, and consequently also to nondeterministic autonomous automata.

From one estimate of H. Wielandt ([3])*, with the aid of (4), it follows directly that

Theorem 1. *If $\mathfrak{A}$ is a primitive automaton, then*

$$d(\mathfrak{A}) \leqslant n^2 - 2n + 3. \tag{5}$$

This interpretation of Wielandt' s estimate is essentially known ([7,8]).

Relying on Wielandt' s estimate, one can also prove the following assertion.

Theorem 2. *If $\mathfrak{A}$ is an imprimitive automaton with index of imprimitivity h, then*

$$d(\mathfrak{A}) \leqslant \frac{1}{h}(n^2 - 2nh + 4h^2). \tag{6}$$

Comparison of the estimates (5), (6) leads to the following conclusion.

Corollary. *For any irreducible automaton $\mathfrak{A}$ with $n \geqslant 5$, estimate (5) holds.*

Estimate (5) is sharp in the class of irreducible (even primitive) automata ([3]).

In the case of a reducible automaton $\mathfrak{A}$, the estimate for $d(\mathfrak{A})$ becomes much worse in comparison with estimate (5).

Lemma. *If the matrix A has the form*

$$A = \begin{pmatrix} U & 0 \\ W & V \end{pmatrix},$$

where U, V are square blocks, then

$$t(A) < t(U) + l(V) + m(p(U), p(V))$$

and $p(A) = m(p(U), p(V))$, where $m(\ ,\ )$ denotes the least common multiple.

On the basis of this lemma one establishes

Theorem 3. *For any automaton $\mathfrak{A}$,*

$$d(\mathfrak{A}) < m(h_1, h_2, \dots, h_r) + \sum_{k=2}^{r} m(h_1, h_2, \dots, h_k) + \sum_{k=1}^{r} \frac{1}{h_k} \left(v_k^2 - 2v_{kh}k + 4h_k^2\right), \tag{7}$$

where $v_1, v_2, \dots, v_r$ are the numbers of states of the irreducible components of the automaton $\mathfrak{A}$, and $h_1, h_2, \dots, h_r$ are their indices of imprimitivity.

Of the numerous consequences of Theorem 3 we note the three most interesting.

Corollary 1. *If all irreducible components of the automaton $\mathfrak{A}$ are primitive, then for $n \geqslant 4$ estimate (5) holds.*

Corollary 2. *For any automaton $\mathfrak{A}$,*

$$d(\mathfrak{A}) < r\left(\frac{n}{r}\right)^r + (n-r)^2 + 5r, \tag{8}$$

where r is the number of irreducible components of the automaton $\mathfrak{A}$.

Corollary 3. *For any automaton $\mathfrak{A}$ with $n \geqslant 6$, the estimate holds*

$$d(\mathfrak{A}) < M(n) + n^2 - 2n + 6, \tag{9}$$

* Which is a refinement of an earlier estimate of H. Frobenius ([2]). Wielandt states his estimate without proof

where $M(n)$ is an arithmetic function defined as the maximum of the expression

$$m(N_1, N_2, \dots, N_s) + \sum_{k=2}^{s} m(N_1, N_2, \dots, N_k)$$

over all systems $(N_1, N_2, \dots, N_s)$ $(s = 2, 3, \dots)$ of natural numbers satisfying the condition

$$\sum_{k=1}^{s} N_k \leq n. \tag{10}$$

An estimate close to (9) was found earlier (but not published) by V. G. Bodnarchuk and A. A. Letichevskii, on the basis of other considerations.

It is not difficult to prove that, although the function $M(n)$ grows faster than any power n^β, it nevertheless grows more slowly than any exponential $e^{\varepsilon n}$ $(\varepsilon > 0)$. These facts were refined by I. V. Ostrovskii, who proved that

$$\ln M(n) \sim \sqrt{n \ln n} \qquad (n \to \infty). \tag{11}$$

Thus, $M(n)$ is the principal term of estimate (9).

Estimate (9) is "almost" exact. Namely, for every n there exists an automaton $\mathfrak{A}^{(0)}$ with n states such that $d(\mathfrak{A}^{(0)}) \geq M_1(n)$, where $M_1(n)$ is an arithmetic function such that $\ln M_1(n) \sim \sqrt{n \ln n}$.

Let us single out some classes of decomposable, generally speaking, automata for which estimate (9) can be substantially improved.

Suppose the number r of components of the automaton $\mathfrak{A}$ is not too large, namely, satisfies the inequality

$$r \leq q\sqrt{\frac{n}{\ln n}} \qquad (q = \text{const},\ q < 2). \tag{12}$$

Then, by virtue of (8) and (11),

$$d(\mathfrak{A}) \leq o(M(n)). \tag{13}$$

The same result is obtained if r is not too small, namely,

$$r \geq n - q\sqrt{n \ln n} \qquad (q = \text{const},\ q < 1). \tag{14}$$

Strengthening requirement (14) to

$$r \geq n - \alpha \ln n \qquad (\alpha = \text{const}), \tag{15}$$

we obtain the power estimate

$$d(\mathfrak{A}) \leq O(n^{\beta}), \tag{16}$$

where $\beta = \max(\alpha + 1, 2)$. Estimate (16) is also valid for $r \leq \beta$. In particular, if $r \geq n - \ln n$ or $r \leq 2$, then

$$d(\mathfrak{A}) \leq O(n^2). \tag{17}$$

In conclusion, let us note that, from a methodological point of view, the present work is related to the investigations ([9], [10]).

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Note: Figure translations are in progress. See original paper for figures.