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## Abstract

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# CRITERIA FOR ONE-SIDED INVERTIBILITY OF FUNCTIONS OF SEVERAL ISOMETRIC OPERATORS AND THEIR APPLICATIONS

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In the present communication we continue the study of the operator-ring approach to the solution and investigation of various classes of singular integral equations, begun by I. Ts. Gokhberg in ($^1$). The results of ($^1$) are generalized here to the case of functions of several isometric operators (or several elements of a ring). As an application, with substantial additions, all the results of ($^2$) are obtained for multidimensional integral equations of Wiener–Hopf type and their discrete analogues.

1. Let $R$ be the integer lattice of vectors $j = (j_0, j_1, \dots, j_n)$, and let $R^+$ ($R^-$) be the subset of $R$ defined by the inequality $j_0 \geqslant 0$ ($j_0 \leqslant 0$). By $F$ we denote the set of all points $\zeta = (\zeta_0, \zeta_1, \dots, \zeta_n)$ of the $(n+1)$-dimensional complex Euclidean space such that $|\zeta_l| = 1$ ($l = 0, 1, \dots, n$).

An $\mathfrak{R}$-ring will mean any normed ring $\mathfrak{A}$ of continuous functions on $F$ having the following properties:

a) if $a(\zeta) \in \mathfrak{A}$, then $|a(\zeta)| \leqslant \|a(\zeta)\|$ ($\zeta \in F$);

b) $\mathfrak{A}$ contains all linear combinations of functions of the form $r_0(\zeta_0) r_1(\zeta_1) \cdots r_n(\zeta_n)$, where $r_k$ ($k = 0, 1, \dots, n$) are rational functions of $\zeta_k$ having no poles on the unit circle $|\zeta_k| = 1$, and the set of these linear combinations is dense in $\mathfrak{A}$.

It is not hard to show that the bicompact space of maximal ideals of every $\mathfrak{R}$-ring is homeomorphic to the set $F$.

By $\mathfrak{A}_+$ ($\mathfrak{A}_-$) we denote the closure of all polynomials of the form $\sum_j a_j \zeta^j$, where

$$\zeta^j = \zeta_0^{j_0} \zeta_1^{j_1} \cdots \zeta_n^{j_n},$$

and $j \in R^+$ ($j \in R^-$). The ring $\mathfrak{A}$ is called decomposing if every function $x(\zeta) \in \mathfrak{A}$ is representable in the form

$$x(\zeta) = x_+(\zeta) + x_-(\zeta), \quad \text{where } x_\pm(\zeta) \in \mathfrak{A}_\pm.$$

If $\mathfrak{A}$ is some ring, then by $O_{\mathfrak{A}}$ we denote the group of all invertible elements of $\mathfrak{A}$.

**Theorem 1.** *Let $\mathfrak{A}$ be some $\mathfrak{R}$-ring. In order that a function $a(\zeta) \in O_{\mathfrak{A}}$ admit the factorization*

$$a(\zeta) = a_-(\zeta)\zeta_0^{\varkappa} a_+(\zeta) \quad (\zeta \in F), \tag{1}$$

*where $a_\pm(\zeta) \in O_{\mathfrak{A}_\pm}$ and $\varkappa$ is some integer, it is necessary and sufficient that the ring $\mathfrak{A}$ be decomposing.*

Let us note that the number $\varkappa$ in formula (1) is determined by the equality

$$\varkappa = \varkappa(a) = \frac{1}{2\pi}\left[\arg a(e^{i\varphi}, \zeta_1, \dots, \zeta_n)\right]_{\varphi=0}^{\varphi=2\pi}.$$

This theorem is a generalization of one proposition from ($^1$). Its proof is based on a theorem of G. E. Shilov (($^3$), § 13).

It is easy to see that the ring $W^{(n)}$ of all functions $a(\zeta)$ $(\zeta \in F)$ expandable in an absolutely convergent series

$$a(\zeta) = \sum_{j \in R} a_j \zeta^j \quad (\zeta \in F), \qquad \sum_{j \in R} |a_j| < \infty,$$

with norm defined by the equality $\|a(\zeta)\| = \sum_{j \in R} |a_j|$, is a decomposing $\mathfrak{R}$-ring.

2. Let $K$ be an arbitrary normed ring with identity $e$, and let $v_0, v_1, \dots, v_n$ be mutually commuting elements of $K$ possessing the following properties:

a) the spectrum of each of the elements $v_l$ $(l = 1, 2, \dots, n)$ coincides with the unit circle;

b) the element $v_0$ has an inverse only on the left, $v_0^{(-1)}$ $(v_0^{(-1)} v_0 = e;\ v_0 v_0^{(-1)} \neq e)$, commuting with the elements $v_1, v_2, \dots, v_n$, and moreover $\|v_0\| = \|v_0^{(-1)}\| = 1$;

c)

$$\left|\sum_j a_j \zeta^j\right| \leq \left\|\sum_j a_j v^j\right\| \quad (\zeta \in F),$$

where $v^j = v_0^{(j_0)} v_1^{j_1} \cdots v_n^{j_n}$, $v_0^{(j_0)} = v_0^{j_0}$ for $j_0 \geq 0$ and $v_0^{(j_0)} = (v_0^{(-1)})^{-j_0}$ for $j_0 < 0$.

Denote by $\hat{K}[v]$ the linear span of the elements $v^j$ $(j \in R)$, by $\hat{K}_+[v]$ $(\hat{K}_-[v])$ the linear span of the elements $v^j$, where $j \in R^+$ $(j \in R^-)$, and by $\hat{K}[v_l]$ $(l = 0, 1, \dots, n)$ the linear span of the elements $v_l^{j_l}$ $(-\infty < j_l < \infty)$. By $K[v]$

($K_+[v], K_-[v]$) we shall denote the closure of the set $\hat{K}[v]$ ($\hat{K}_+[v], \hat{K}_-[v]$). To each element $r = \sum_j a_{jv}^j \in \hat{K}[v]$ we associate the function $r(\zeta) = \sum_j a_j \zeta^j$ ($\zeta \in F$). By property c) we have $|r(\zeta)| \leq \|r\|$. Extending this correspondence by continuity, we associate with each element $a \in K[v]$ a continuous function $a(\zeta)$ ($\zeta \in F$), and moreover $|a(\zeta)| \leq \|a\|$. The ring of all functions $a(\zeta)$ corresponding to elements $a \in K[v]$, with the norm defined by $\|a(\zeta)\| = \|a\|$ and with the usual operations, will be denoted by $K[\zeta]$. $K[\zeta]$ is an $\mathfrak{R}$-ring.

**Theorem 2.** Let $a \in K[v]$ and

$$a(\zeta) \neq 0 \quad (\zeta \in F). \tag{2}$$

If $\chi = \chi(a) > 0$, then the element $a$ is invertible in $K$ only on the left; if $\chi < 0$, it is invertible in $K$ only on the right; and if $\chi = 0$, the element $a$ has a two-sided inverse in $K$. If the ring $K[\zeta]$ is decomposing and equality (1) gives a factorization of the function $a(\zeta)$, then the element $a^{(-1)}$, inverse to $a$ on the corresponding side, is determined by the formula $a^{(-1)} = a_+^{-1} v_0^{-\chi} a_-^{-1}$, where $a_\pm \in K_\pm[v]$ are the elements to which the functions $a_\pm(\zeta)$ correspond.

If condition c) is replaced by the stronger condition:

c′) for any elements $r_l \in \hat{K}[v_l]$ ($l = 0, 1, \dots, n$) the equality

$$\|r_0 r_1 \cdots r_n\| = \|r_0\| \cdot \|r_1\| \cdots \|r_n\|$$

holds,

then Theorem 2 admits the following converse.

**Theorem 3.** In order that an element $a \in K[v]$ have an inverse in $K$ at least on one side, it is necessary that condition (2) be satisfied. If condition (2) is not satisfied, then the element $a$ is a two-sided generalized zero divisor.

For the case $n = 0$ these theorems were proved in (1). The proof of Theorem 2 is analogous to the proof of the corresponding theorem from (1), while the proof of Theorem 3 differs substantially from the proof of the corresponding proposition from (1) and is based on the following lemma.

**Lemma 1.** If conditions a), b), and c′) are satisfied, then for every point $\zeta^{(0)} \in F$ there exist sequences of vectors $s_\pm^{(m)} \in K_\pm[v]$ ($m = 1, 2, \dots$) such that for any element $x \in K[v]$

$$\lim_{m\to\infty} \|x s_+^{(m)}\| = \lim_{m\to\infty} \|s_-^{(m)} x\| = |x(\zeta^{(0)})|.$$

3. By $d(R^+)$ we shall denote one of the Banach spaces $l_p(R^+)$ ($p \geq 1$), $m(R^+)$, $c(R^+)$, and $c_0(R^+)$ of numerical sequences $\xi = \{\xi_j\}_{j\in R^+}$ (2).

In the space $d(R^+)$ we introduce the shift operators $V_0, V_1, \dots, V_n$, defined by the equalities:

$$V_0\{\xi_j\} = \{\eta_j\}, \quad \text{where } \eta_{0,j_1,\dots,j_n} = 0 \text{ and } \eta_{j_0+1,j_1,\dots,j_n} = \xi_{j_0,j_1,\dots,j_n},$$

$$V_k\{\xi_j\} = \{\xi_{j_0, j_1, \dots, j_{k-1}, j_k - 1, j_{k+1}, \dots, j_n}\} \quad (k = 1, 2, \dots, n).$$

It can be verified that the linear isometric operators $V_0, V_1, \dots, V_n$, considered as elements of the ring $K$ of all linear bounded operators acting in the space $d(R^+)$, satisfy all the conditions of item 2. Let us note that the system of equations

$$\sum_{j \in R^+} a_{k-j} \xi_j = \eta_k \quad (k \in R^+), \tag{3}$$

which is a discrete analogue of the multidimensional Wiener-Hopf equation, can be written in the form $\left(\sum_{j\in R} a_j V^j\right) \xi = \eta$, where $\xi = \{\xi_k\} \in d(R^+)$ and $\eta = \{\eta_k\} \in d(R^+)$. Applying the results of item 2 to the operator $A = \sum_{j\in R} a_{jV}^{j}$, we obtain the following propositions.

**Theorem 4.** In order that the system of equations (3) be normally solvable and that the corresponding homogeneous system of equations have only the zero solution, it is necessary and sufficient that conditions (2) hold and that $\chi = \chi(a) \geqslant 0$. If $a(\zeta) \neq 0$ $(\zeta \in F)$, $\chi \geqslant 0$, the solvability conditions are satisfied and the equality

$$a^{-1}(\zeta) = a_+(\zeta) \zeta_0^{-\chi} a_-(\zeta) \quad \left(a_\pm(\zeta) \in O^{(n)}_{W_\pm}\right) \tag{4}$$

gives a factorization of the function $a^{-1}(\zeta)$, then the solution of the system (3) has the form

$$\xi_j = \sum_{k \in R^+} a_{jk} \eta_k \quad (j \in R^+),$$

where

$$a_{jk} = \sum_{r \in R^+} a^+_{j-r} a^-_{r-\bar{\chi}-k}, \qquad \bar{\chi} = (\chi, 0, 0, \dots, 0) \in R,$$

and the numbers $a_j^\pm$ are determined from the equality

$$a_\pm(\zeta) = \sum_{l \in R^\pm} a_l^\pm \zeta^l.$$

**Theorem 5.** In order that the system (3) be solvable for an arbitrary right-hand side, it is necessary and sufficient that conditions (2) hold and that $\chi = \chi(a) \leqslant 0$. If $a(\zeta) \neq 0$ $(\zeta \in F)$, $\chi < 0$, and equality (4) gives a factorization of the function $a^{-1}(\zeta)$, then:

a) one of the solutions of the system (3) has the form

$$\xi_j = \sum_{k \in R^+} a_{j+\bar{\chi},\, k} \eta_k \quad (j \in R^+),$$

where $\bar{\chi}, a_j^{\pm}$ are the same as in Theorem 4;

b) the subspace $\mathfrak{Z}$ of all solutions of the corresponding homogeneous system is finite-dimensional and can be represented in the form

$$\mathfrak{Z} = L + V_0 L + \cdots + V^{-\chi-1} L,$$

where $L = A_+ L_1$, $A_+ = \sum_{j \in R^+} a_j^+ V^j$, and $L_1$ is the subspace of $d(R^+)$ consisting of all sequences $\xi = \{\xi_j\}$ for which $\xi_j = 0$ when $j_0 \neq 0$.

Theorems 4 and 5, on the one hand, refine the results of paper ($^2$), and, on the other hand, generalize the theorems of I. Ts. Gokhberg ($^1$), proved for the case $n = 0$. In the proof of Theorem 4 an essential role is played by the following:

**Lemma 2.** For each point $\zeta^{(0)} \in F$ there exists a sequence of vectors $f_m \in d(R^+)$ $(m = 1, 2, \ldots)$ such that for every operator $A = \sum_j a^j_{jV} \in \hat{K}[V]$ the equality

$$\lim_{m \to \infty} \|A f_m\| = \left| \sum_j a_j \left(\zeta^{(0)}\right)^j \right|$$

holds.

4. Let $d(R^+) = l_1(R^+)$, and let $\Phi$ be the set of $n$-dimensional vectors $\mu = (\zeta_1, \zeta_2, \ldots, \zeta_n)$ such that $|\zeta_1| = |\zeta_2| = \cdots = |\zeta_n| = 1$. To each element $\xi = \{\xi_j\} \in d(R^+)$ assign the sequence of functions $\chi =$

$$= \{x_r(\mu)\}_{r=0}^{\infty} \quad (\mu \in \Phi), \quad \text{where}$$

$$x_r(\mu) = \sum_{j_1, \ldots, j_n} \xi_{r, j_1, \ldots, j_n} \xi_1^{j_1} \cdots \xi_n^{j_n} \quad (\mu \in \Phi).$$

If we put $\|x\| = \|\xi\|$, then the space $\hat{E}$ of these sequences $x$ is isometric to the space $d(R^+)$.

The system of equations (3) can be written in the form

$$\sum_{k=0}^{\infty} a_{j-k}(\mu) x_k(\mu) = y_j(\mu) \quad (j = 0, 1, \ldots;\ \mu \in \Phi), \tag{5}$$

where

$$a_r(\mu) = \sum_{j_1,\dots,j_n} a_{r,j_1,\dots,j_n} \xi_1^{j_1} \cdots \xi_n^{j_n};$$

$$x_r(\mu) = \sum_{j_1,\dots,j_n} \xi_{r,j_1,\dots,j_n} \xi_1^{j_1} \cdots \xi_n^{j_n}, \quad y_r(\mu) = \sum_{j_1,\dots,j_n} \eta_{r,j_1,\dots,j_n} \xi_1^{j_1} \cdots \xi_n^{j_n},$$

i.e., in the form of a one-dimensional Wiener-Hopf system, but with functional coefficients.

**Theorem 6.** If $a(\xi) \neq 0$ $(\xi \in F)$ and $x = x(a) = 0$, then, starting from some $s$ $(s \geq s_0)$, each of the systems of equations

$$\sum_{k=0}^{s} a_{j-k}(\mu) x_k(\mu) = y_j(\mu) \quad (j = 0, 1, \dots, s;\ \mu \in \Phi)$$

has a unique solution $\{x_k^{(s)}(\mu)\}_{k=0}^{s}$. As $s \to \infty$, the vector

$$x^{(s)}(\mu) = \{x_0^{(s)}(\mu), x_1^{(s)}(\mu), \dots, x_s^{(s)}(\mu), 0, 0, \dots\}$$

converges in norm to the solution of system (5).

This theorem for the case $n = 0$ was proved by G. Baxter ($^4$).

5. Denote by $t = (t_0, t_1, \dots, t_n)$ the vectors of the $(n+1)$-dimensional real Euclidean space $E$, and by $E^+$ $(E^-)$ the half-space of $E$ defined by the inequality $t_0 \geq 0$ $(t_0 \leq 0)$. By $D(E^+)$ (see ($^2$)) we shall denote one of the spaces of complex-valued functions $L_p(E^+)$ $(p \geq 1)$ and $C_0(E^+)$.

**Theorem 7.** In order that the multidimensional Wiener-Hopf integral equation

$$\varphi(t) - \int_{E^+} k(t-s)\varphi(s)\, ds = f(t) \quad (t \in E^+;\ k(t) \in L_1(E)) \tag{6}$$

have a unique solution $\varphi(t) \in D(E^+)$ for every right-hand side $f(t) \in D(E^+)$, it is necessary and sufficient that

$$1 - \int_E k(t) e^{i(\lambda,t)}\, dt \neq 0 \quad (\lambda \in E). \tag{7}$$

The sufficiency of condition (7) was established in ($^2$). The proof of the necessity of this condition is based on a lemma analogous to Lemma 2.

All the results presented admit a generalization to the even multidimensional integral equation, its transpose, and their discrete analogues.

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*Note: Figure translations are in progress. See original paper for figures.*

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