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Abstract

Full Text

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AN INHOMOGENEOUS RIEMANN BOUNDARY-VALUE PROBLEM WITH INFINITE INDEX

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MATHEMATICS

1°. Let, in the complex z-plane, a domain D be given whose boundary L is the ray $1 \leq t \leq \infty$. We consider in the domain D the Riemann boundary-value problem with infinite index

$$\Phi^+(t) = G(t)\Phi^-(t) + g(t) \tag{1}$$

under the following assumptions:

1) $\arg G(t) = 2\pi\varphi(t)t^\rho, \quad -1 < \varphi(1) \leq 0, \quad \varphi(\infty) \neq 0, \quad 0 < \rho < \infty,$

(2)

where $\varphi(t) \in H(\mu)$, i.e.

$$|\varphi(t_1) - \varphi(t_2)| < A\left|\frac{1}{t_1} - \frac{1}{t_2}\right|^\mu, \quad t_k \in L, \quad A, \mu = \text{const}, \tag{3}$$

and

$$\frac{\rho}{\rho+1} < \mu \leq 1; \tag{4}$$

2) $\ln|G(t)|,\ g(t) \in H(\lambda) \quad (0 < \lambda \leq 1);$

(5)

3) $g(1) = g(\infty) = 0.$

(6)

The case $g(t) \equiv 0$ (the homogeneous problem) was considered in ([5,6]). The inhomogeneous problem was solved in ([5]) for the case $0 < \rho < {}^1/_2,\ \mu > \rho,\ \varphi(\infty) > 0$.

2°. Definition 1. Let a set of points $\{z_n\}$ be given in the domain D, with $\lim_{n\to\infty} z_n = \infty$. Then, if for all ψ_1, ψ_2 such that

$$-2\pi < \psi_1 < \psi_2 < 2\pi, \quad |\psi_1 - \psi_2| < 2\pi, \quad \psi_k \in N,$$

where N is at most countable, for some $\sigma > 0$ there exists the limit

$$\lim_{r\to\infty} \frac{1}{r^\sigma} \sum_{\substack{\psi_1 < \arg z_n \le \psi_2 \\ |z_n| \le r}} \arg z_n = \nu(\psi_1, \psi_2) \ne \infty \tag{7}$$

and the asymptotic estimate

$$\sum_{\substack{-\pi < \arg z_n \le \pi \\ |z_n| \le r}} |\arg z_n| < K r^\sigma \quad (K > 0,\ K = \text{const})$$

holds, then we shall say that the set $\{z_n\}$ has, in the domain D, an **argument density**. In doing so we adopt, by definition,

$$\nu(\psi, \psi) = 0; \qquad \nu(\psi_1, \psi_2) = -\nu(\psi_2, \psi_1), \quad \text{if } \psi_1 > \psi_2,$$

$$\nu(\psi_1, \psi_2) = \nu(\psi_1, +0) + \nu(+0, \psi_2), \quad \text{if } |\psi_1 - \psi_2| > 2\pi.$$

By the **argument density** we shall mean the function $\nu(\psi) = \nu(\psi_0, \psi)$, determined up to an additive constant for fixed ψ_0, extended at the points of the set N by right-continuity.

The concept of argument density is closely connected with the concept of angular density introduced by B. Ya. Levin (([2]), p. 118). We note that the quantity $\sum_{|z_n| \le r} \sin \arg z_n$, related to the sum in (7), was considered by R. Nevanlinna ([3]).

We shall call the following function the **indicator** of a function $f(z)$, regular and of order $\sigma > 0$ in the angle $\alpha < \arg z < \beta$ (( ), p. 1247; ( ), p. 209):

$$h_f(\theta) = \varlimsup_{r\to\infty} r^{-\sigma} \ln \left| f\left(re^{i\theta}\right) \right| \qquad (\alpha < \theta < \beta).$$

Definition 2. A function $f(z)$, regular and of order $\sigma > 0$ in the angle $\alpha < \arg z < \beta$, continuous for $\alpha \le \arg z \le \beta$ (for $z \ne \infty$), is called a function of **completely regular growth in the closed angle** $\alpha \le \arg z \le \beta$, if the function $r^{-\sigma} \ln |f(re^{i\theta})|$ tends to $h_f(\theta)$ uniformly for $\alpha \le \theta \le \beta$, as r tends to $+\infty$, except for values from a set E, common to all θ, of zero relative measure (([2]), pp. 127, 182).

We shall call $f(z)$ a function of **completely regular growth** (of order $\sigma > 0$) in the **open angle** $\alpha < \arg z < \beta$, if it has completely regular growth in each angle $\alpha + \varepsilon \leq \arg z \leq \beta - \varepsilon$ $(\varepsilon > 0)$ and if, asymptotically,

$$\sup_{|z| \leq r} |f(z)| < \exp(K r^{\sigma}) \qquad (K > 0).$$

Denote by B_σ the class of functions regular, of finite order $\sigma > 0$, and of completely regular growth in the domain D.

3°. Consider problem (1) for the case of a plus-infinite index, i.e., when $\varphi(\infty) > 0$. We shall solve it in the class B_σ, where $0 < \sigma < \min(\rho, \frac{1}{2})$. (For $\sigma > \min(\rho, \frac{1}{2})$ there are no solutions.) Denote by $\Psi(z)$ the solution of the corresponding homogeneous problem (1):

$$\Psi^+(t) = -G(t)\Psi^-(t). \tag{8}$$

Theorem 1. *Let $\Phi_0(z)$ be some bounded solution of problem (1) (with $\varphi(\infty) > 0$) of order not less than σ_0, $0 < \sigma < \sigma_0 < \min(\rho, \frac{1}{2})$. Then the general solution of this problem in the class B_σ has the form*

$$\Phi(z) = \Phi_0(z) + \Psi(z), \tag{9}$$

where $\Psi(z)$ is the general solution of the homogeneous problem (8) in the class B_σ. Moreover, the indicators of the functions $\Phi(z)$ and $\Psi(z)$ in the interval $0 < \theta < 2\pi$ coincide.

The general solution of the homogeneous problem was found in ( ). It remains to find $\Phi_0(z)$. Applying the method of F. D. Gakhov (([1]), p. 117), one can obtain:

$$\Phi_0(z) = \frac{\Psi_0(z)}{2\pi i} \int_1^{\infty} \frac{g(x)\,dx}{\Psi_0^+(x)(x - z)}, \tag{10}$$

where $\Psi_0(z)$ is some solution of the homogeneous problem (8). In order that the integral converge and that $\Phi_0(z)$ be bounded and of order not less than σ_0, it is sufficient to find such a $\Psi_0(z)$ that the following conditions be satisfied:

a) $\Psi_0(z)$ is bounded in D;

b) in any angle $\varepsilon \leq \arg z \leq 2\pi - \varepsilon$, for $|z| = r > r_\varepsilon$, the estimate

$$\max_{\varepsilon \leq \theta \leq 2\pi - \varepsilon} \left|\Psi_0\left(re^{i\theta}\right)\right| < \exp\left(-K_\varepsilon r^{\sigma_0}\right) \qquad (K_\varepsilon > 0);$$

c) $g(t)/\Psi_0^+(t) \in H(\tau) \quad (0 < \tau \leq 1)$, $\lim_{t\to\infty} g(t)/\Psi_0^+(t) = 0$.

One of the simplest functions subject to conditions a)–c) has the form*

$$\Psi_0(z) = \exp\left[\frac{z}{2\pi i}\int_0^\infty \frac{\ln G(x) - 2\pi i n_{\Psi_0}(x)}{x(x-z)}\,dx\right]\prod_{n=1}^{\infty}\frac{\left(1-\frac{z}{r_n}e^{-ir_n^{-\rho}}\right)\left(1+\frac{z}{s_n}\right)}{\left(1-\frac{z}{r_n}\right)\left(1-\frac{z}{s_n}\right)}, \tag{11}$$

where

$$s_n = \left(\frac{2n-1}{2\cos\sigma_0\pi}\right)^{1/\sigma_0}, \quad 1 \le r_1 \le r_2 \le \cdots,$$

and the number $n_1(r)$ of points r_n on the interval $[1, r]$ is determined by the equality

$$n_1(r) = \max\left\{\left[\max_{1\le x\le r}\left\{\varphi(x)x^\rho - x^{\sigma_0} + \frac{1}{2}\right\}\right], 0\right\}^{**},$$

while $n_{\Psi_0}(r)$ is the total number of points $s_n, r_n \in [0, r]$ ((6), p. 16).

Let us formulate the final result.

Theorem 2. The general solution of the nonhomogeneous problem (1) in the class B_σ, $0 < \sigma < \min(\rho, 1/2)$, has the form

$$\Phi(z) = \frac{\Psi_0(z)}{2\pi i}\int_1^\infty \frac{g(x)\,dx}{\Psi_0^+(x)(x-z)} +$$

$$+cz^m \exp\left[\frac{z}{2\pi i}\int_0^\infty \frac{\ln G(x) - 2\pi i\tilde{n}(x)}{x(x-z)}\,dx\right] \times \prod_{n=1}^{\infty}\frac{1 - z/z_n}{1 - z/|z_n|}, \tag{12}$$

where $\Psi_0(z)$ is defined by formula (11), $\tilde{n}(r)$ is the number of points z_n in $0 < |z| \le r$, and the following conditions are satisfied:

I. The set $\{z_n\}$ has angular σ-density $\nu(\psi)$.

II. There exists the finite limit

$$\lim_{r\to\infty}\frac{\sigma^2}{r^\sigma}\int_0^r \frac{dt}{t}\int_0^t \frac{\frac{1}{2\pi}\arg G(x) - \tilde{n}(x)}{x}\,dx = \gamma \ge 0,$$

where, if $\gamma = \gamma(\sigma) = 0$, then $\gamma(\sigma - \varepsilon) = +\infty$ for any $\varepsilon > 0$.

III. The integral

$$\int_0^\infty \frac{1}{x^2}\left[\frac{1}{2\pi}\arg G(x) - \tilde{n}(x)\right] dx$$

converges.

IV. For $t > t_\infty,\ t \neq |z_n|$, the estimate

$$t\int_0^\infty \frac{\frac{1}{2\pi}\arg G(x) - \tilde{n}(x)}{x(x-t)}\,dx + m\ln t + \sum_{n=1}^{\infty}\ln\left|\frac{z_n - t}{|z_n| - t}\right| < C_\Phi = \text{const}$$

holds.

In this case the indicator of the solution is expressed by the formula $(0 < \theta < 2\pi)$

$$h_\Phi(\theta) = \frac{\pi}{\sin\pi\sigma}\left[\int_{-\pi}^{\pi}\alpha(\psi,\theta)\,d\nu(\psi) - \gamma\cos\sigma(\theta-\pi)\right],$$

where it is set that

$$\alpha(\psi,\theta) = \begin{cases} \psi^{-1}\left[\cos\sigma(|\theta-\psi|-\pi) - \cos\sigma(\theta-\pi)\right], & 0 < \psi \le \pi, \\ \psi^{-1}\left[\cos\sigma(|\theta-\psi-2\pi|-\pi) - \cos\sigma(\theta-\pi)\right], & -\pi \le \psi < 0, \\ \sigma\sin\sigma(\theta-\pi), & \psi = 0. \end{cases}$$

Corollary. Problem (1) has in the class B_σ, $0 < \sigma < \min(\rho, 1/2)$, an infinite set of linearly independent solutions.

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* We assume that $G(t) = 1,\ g(t) = 0$ for $0 \le t < 1$.

** $[a]$ denotes the integer part of the real number a.

Remark. The convergence of the infinite product and of the second of the integrals in equality (12) follows respectively from I and III.

4°. Let us now consider problem (1) for minus-infinite index.

Theorem 3. Let assumptions (2)–(6) be satisfied, with $\varphi(\infty) < 0$, and let $\widetilde{\Psi}(z)$ be meromorphic in the domain D and such that:

1) $\widetilde{\Psi}^+(t) = G(t)\widetilde{\Psi}^-(t) \quad (1 < t < \infty)$;
2) $g(t)/\widetilde{\Psi}^+(t) \in H$;
3) $|\widetilde{\Psi}^\pm(t)| < \text{const} \quad (1 \le t < \infty)$;
4) on some sequence of circles $|z| = k_n$ $(k_n \to \infty)$

$$\frac{1}{M} < \sup_{n=1,2,\dots} \max_{|z|=k_n} \left|\widetilde{\Psi}(z)\right| < M < \infty,$$

where $M = \text{const}$.

Then, for the solvability of problem (1) in the class of bounded functions, it is necessary and sufficient that at all poles z_n of the function $\widetilde{\Psi}(z)$, the number of which is known to be infinite, the equalities

$$\int_1^\infty \frac{g(x)\,dx}{\widetilde{\Psi}^+(x)(x - z_n)} = 0 \qquad (n = 1, 2, \dots) \tag{13}$$

hold.

If (13) is fulfilled, the solution is unique and is expressed by the formula

$$\Phi(z) = \frac{\widetilde{\Psi}(z)}{2\pi i} \int_1^\infty \frac{g(x)\,dx}{\widetilde{\Psi}^+(x)(x - z)}. \tag{14}$$

One of the simplest functions subject to conditions 1)–4) has the form

$$\widetilde{\Psi}(z) = \exp\left[\frac{z}{2\pi i} \int_1^\infty \frac{\ln G(x) + 2\pi i p(x)}{x(x - z)}\,dx\right] \prod_{n=1}^\infty \frac{1 - z/|z_n|}{1 - z/z_n},$$

where $p(r)$ denotes the number of poles z_n of the function $\widetilde{\Psi}(z)$ in $0 < |z| \le r$, and

$$z_n = r_n e^{i r_n^{-\rho}}, \qquad p(r) = \left[\max_{1 \le x \le r} \left\{\frac{1}{2} - \varphi(x) x^\rho\right\}\right].$$

Remark. For $0 < \rho < 1/2$, (14) can be simplified to the form

$$\Phi(z) = \frac{X(z)}{2\pi i} \int_1^\infty \frac{g(x)\,dx}{X^+(x)(x - z)}, \qquad \text{where} \quad X(z) = \exp\left[\frac{z}{2\pi i} \int_1^\infty \frac{\ln G(x)\,dx}{x(x - z)}\right]$$

(if (13) is fulfilled); in this case the integrals are certainly convergent.

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Note: Figure translations are in progress. See original paper for figures.