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Abstract

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ON REPRESENTATIONS OF FINITE GROUPS OVER QUADRATIC RINGS

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Let G be a finite group, and let L be a commutative ring with identity. By representations of the group G over the ring L we shall mean representations of the group G by matrices over L. Denote by $n(G, L)$ the number of indecomposable L-representations of the group G. The problem is posed:

(A) For which pairs (G, L) is the number $n(G, L)$ finite?

This problem has been solved for the following rings L: 1) the ring of rational integers ([1–5]); 2) the ring of integers of the field $R_p(\xi)$ (R_p is the field of p-adic numbers, $\xi^d = 1$) ([6–8]); 3) the ring of residue classes modulo m ([9]).

In addition, the author ([6–7]) and independently Dade ([8]) proved the following lemma:

Lemma 1. *Let F be a finite extension of the field of p-adic numbers R_p, and let T be the ring of integers of the field F (with respect to the norm in F). If a p-group H has at least 4 irreducible representations over the field F, then the number $n(H, T)$ is infinite.*

By virtue of Lemma 1, the solution of problem (A) for $L = T$ is easily reduced to the case where G is a cyclic p-group of order p^m ($m \leq 2$); moreover, for $p > 3$ the field F does not contain a primitive p-th root of unity. The latter problem is closely connected with the study of modules over the ring $T[\eta]$ ($\eta^{p^\alpha} = 1;\ 1 \leq \alpha \leq 2$). In the studied cases of problem (A) (for numerical rings of characteristic 0), the ring $L[\eta]$ was Dedekind, which made it possible to use the well-known theorem of Steinitz ([10]) on modules over Dedekind rings*.

Let Q be a quadratic extension of the field of rational numbers R; let K be the ring of algebraic integers of the field Q; let Q' be a quadratic extension of the field of p-adic numbers R_p, and let K' be the ring of integers of the field Q'. In the present note, problem (A) is solved for the rings K and K'. In this case the rings $K[\eta]$ and $K'[\eta]$, generally speaking, will not be Dedekind.

Lemma 2. *Let H be a cyclic p-group of order p^m ($1 \leq m \leq 2$), let ε be a primitive p-th root of unity, and let T be the ring of integers of a finite extension F of the field R_p. If the group H has exactly three irreducible F-representations* *and the ideal $I = (1-\varepsilon)$ of the ring $T[\varepsilon]$ is not principal, then the number $n(H,T)$ is infinite.*

It follows from Lemma 2 that the following hypothesis of Dade ([8]) is false: if a p-group H has no more than three irreducible F-representations, then $n(H,T) < \infty$. (An example is easily constructed.)

Lemma 3 ([11]). *Let F_1 and F_2 be finite extensions of the field of p-adic numbers R_p, with $F_1 \subset F_2$; let T_i be the ring of integers of the field F_i $(i = 1,2)$, and let Γ_1 and Γ_2 be two T_1-representations of a finite group G. The representations Γ_1 and Γ_2 are T_1-equivalent if and only if they are T_2-equivalent.*

Next, let Q' be a quadratic extension of the field of p-adic numbers R_p, and let K' be the ring of integers of the field Q'. If Q' is an unramified extension of the field R_p, then problem (A) for the ring K' is solved in exactly the same way as for the ring of integers of the p-adic numbers ([1]).

Let now

$$Q' = \begin{cases} R_p(\sqrt{p}), & \text{if } p \equiv 1 \pmod 4, \\ R_p(\sqrt{p\omega}), & \text{if } p \equiv -1 \pmod 4, \end{cases} \tag{1}$$

* Problem (A) can be solved for any numerical ring L for which $L[\eta]$ is a Dedekind ring ($\eta^{p^\alpha} = 1;\ 0 \le \alpha \le 2$).

(ω is a primitive root of degree $p-1$ of unity). In this case the ring $K'[\varepsilon]$ will be the ring of integers of the field $Q'(\varepsilon)$ ($\varepsilon^p = 1$). It is easy to see that the circle-division polynomial $\Phi_p(x)$ of order p decomposes over the field of the form (1) into the product of two irreducible polynomials of the same degree $s = (p-1)/2$: $\Phi_p(x) = f_1(x) f_2(x)$. Hence it follows that the cyclic group $H = (a)$ of order p has exactly 3 irreducible representations over the ring K' of integers of the field Q' of the form (1), namely:

$$a \to 1, \qquad a \to \tilde{\varepsilon}_i \quad (i = 1,2), \tag{2}$$

where $\tilde{\varepsilon}_i$ is the matrix corresponding to the operator of multiplication by the primitive root ε_i of degree p of 1 in the K'-basis $1, \varepsilon_i, \dots, \varepsilon_i^{s-1}$ of the ring $K'[\varepsilon_i]$ ($f_i(\varepsilon_i) = 0;\ i = 1,2;\ s = (p-1)/2$).

Any K'-representation of the group $H = (a)$ with irreducible components $\tilde{\varepsilon}_1$ and $\tilde{\varepsilon}_2$ can be written in the form

$$a \to \begin{pmatrix} \tilde{\varepsilon}_1 & \langle\delta\rangle \\ 0 & \tilde{\varepsilon}_2 \end{pmatrix}, \tag{3}$$

where $\langle\delta\rangle$ is a matrix over K' in which all columns except the last are zero, and the last consists of the coordinates of the element $\delta \in K'[\varepsilon_1]$ in the K'-basis $1, \varepsilon_1, \dots, \varepsilon_1^{s-1}$ of the ring $K'[\varepsilon_1]$.

Theorem 1. *Let Q' be a quadratic field of the form (1), K' the ring of integers of the field Q', and $P = (u)$ a prime ideal of the ring $K'[\varepsilon_1]$. All indecomposable K'-representations of the cyclic group H of order p are exhausted by the following representations:*

$$1)\quad 1; \qquad 2)\quad \tilde{\varepsilon}_i \ (i = 1, 2); \qquad 3)\quad \begin{pmatrix} \tilde{\varepsilon}_i & \langle 1 \rangle \\ 0 & 1 \end{pmatrix} \ (i = 1, 2);$$

$$4)\quad \begin{pmatrix} \tilde{\varepsilon}_1 & \langle u^r \rangle \\ 0 & \tilde{\varepsilon}_2 \end{pmatrix}; \qquad 5)\quad \begin{pmatrix} \tilde{\varepsilon}_1 & \langle u^r \rangle & \langle 1 \rangle \\ 0 & \tilde{\varepsilon}_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \left(r = 0, 1, \dots, \frac{p-3}{2} \right);$$

$$6)\quad \begin{pmatrix} \tilde{\varepsilon}_1 & \langle u^j \rangle & 0 \\ 0 & \tilde{\varepsilon}_2 & \langle 1 \rangle \\ 0 & 0 & 1 \end{pmatrix} \left(j = 0, 1, \dots, \frac{p-1}{2} \right);$$

$$7)\quad \begin{pmatrix} \tilde{\varepsilon}_1 & \langle u^i \rangle & 0 & \langle 1 \rangle \\ 0 & \tilde{\varepsilon}_2 & \langle 1 \rangle & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \left(i = 1, 2, \dots, \frac{p-3}{2};\ p > 3 \right)$$

(see the notation (2) and (3)).[1]

Since the cyclic group H of order p^2 has at least 4 irreducible representations over the field Q' of the form (1), it follows, by Lemma 1, that the number $n(H, K')$ in this case is infinite.

Let, further,

$$Q' = \begin{cases} R_p(\sqrt{p}), & \text{if } p \equiv -1 \pmod 4, \\ R_p(\sqrt{p\omega}), & \text{if } p \equiv 1 \pmod 4 \end{cases} \tag{4}$$

(ω is a primitive root of degree $p-1$ of 1); K' is the ring of integers of the field Q'; ε is a primitive root of degree p of 1; $\pi = (\varepsilon - 1)$;

$t = \sqrt{p}$ or $\sqrt{p\omega}$; $I_0 = K'[\varepsilon]$, and I_r is an ideal of the ring $K'[\varepsilon]$ ($r = 1, \dots, s;\ s = (p-1)/2$) with K'-basis $v_1^{(r)}, \dots, v_{p-1}^{(r)}$: $v_1^{(r)} = t;\ v_i^{(r)} = \pi^{r-2+i}$ ($i = 2, \dots, d;\ d = p - r$); $v_{d+j}^{(r)} = t\pi^j$ ($j = 1, \dots, r-1$).

Lemma 4. A group H of order p has over the ring K' of integers of a field Q' of the form (4) exactly $(p+3)/2$ indecomposable representations, and all of them, except the identity representation, are realized in the ideals I_r of the ring $K'[\varepsilon]$ ($r = 0, 1, \dots, s$).

[1] We note that the indecomposable representations of the cyclic group of order p over the ring of integers K' of a field of the form (1) are constructed according to the type of indecomposable representations of the cyclic group of order p^2 over the ring of integers of the p-adic numbers ([1]).

Denote by $\widetilde{T}_r$ the matrix corresponding to the operator of multiplication by ε in the K'-basis $v_1^{(r)}, \dots, v_{p-1}^{(r)}$ of the ideal I_r ($r = 1, \dots, s;\ s = (p-1)/2$).

Theorem 2. Let K' be the ring of integers of a field Q' of the form (4). A cyclic group of order p has only the following indecomposable representations over the ring K':

$$1)\ 1; \qquad 2)\ \tilde{\varepsilon}; \qquad 3)\ \widetilde{T}_r\ (r = 1, \dots, s); \qquad 4)\ \begin{pmatrix} \tilde{\varepsilon} & \langle t^i \rangle \\ 0 & 1 \end{pmatrix}\ (i = 0, 1);$$

$$5)\ \begin{pmatrix} \widetilde{T}_r & A \\ 0 & 1 \end{pmatrix}, \quad \text{where } A = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad (r = 1, 2, \dots, s;\ s = (p-1)/2);$$

$$6)\ \begin{pmatrix} \widetilde{T}_r & B \\ 0 & 1 \end{pmatrix}, \quad \text{where } B = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad (r = 1, 2, \dots, s-1;\ p > 3);$$

$$7)\ \begin{pmatrix} \widetilde{T}_r & C \\ 0 & E \end{pmatrix}, \quad \text{where } E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} E \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad (r = 1, \dots, s-1;\ p > 3)$$

(see notations (2) and (3)). The number of representations 1)–7) is equal to $2p$.

Lemma 4 and Theorem 2 are proved using Lemma 3.

The representations of type 7) show that indecomposable K'-representations of the abelian p-group H, even with two irreducible Q'-components, generally speaking, are not realized in ideals of the group ring $K'H$ of the group H over the ring K'. Let us note one more interesting circumstance: if S_{pH} is the group ring of an abelian group H over the ring of integers of the p-adic numbers S_p, then for two indecomposable S_{pH}-modules M_1 and M_2, $\mathrm{Ext}(M_1, M_2) = 0$ if M_1 and M_2 are isomorphic as R_{pH}-modules (R_p is the field of p-adic numbers). It turns out that this fact does not hold in general for indecomposable $K'H$-modules.

It follows from Lemma 2 that the cyclic group of order p^2 possesses infinitely many indecomposable representations over the ring of integers of a field Q' of the form (4). If Q' is a ramified quadratic extension of the field of 2-adic numbers R_2, then it also follows from Lemma 2 that the cyclic group of order 4 has infinitely many indecomposable representations over the ring of integers of the field Q'.

Theorem 3. Let G be a finite group; Q' a quadratic extension of the field of p-adic numbers R_p; K' the ring of integers of the field Q'. The number $n(G, K')$ is finite if and only if the Sylow p-subgroup H of the group G is cyclic, of order not divisible by p^3, and, moreover, if $(H:1) = p^2$, then Q' is an unramified extension of the field R_p.

Theorem 4. Let G be a finite group, Q a quadratic extension of the field of rational numbers R, and K the ring of all algebraic integers of the field Q. The number $n(G, K)$ is finite if and only if, for every prime number $p \mid (G:1)$, the Sylow p-subgroup of the group G is cyclic of order p^m $(m \leq 2)$, and, if $m = 2$, then the number p is not ramified in the field Q.

The last theorem is proved using the results of the paper ([5]).

Let P be a prime ideal of the ring K $(p \equiv 0 \pmod{P})$; K_P the ring of P-integral numbers of the field Q; K' the ring of integers of the P-adic completion of the field Q, and Z_p the ring of p-integral rational numbers. It follows from (12–14) that a Z_p-representation of a p-group is indecomposable over Z_p if and only if it is indecomposable over the ring of integers of the p-adic numbers. It turns out that already for representations of the cyclic group H of order p over the ring K_P this assertion is no longer valid. One can give an example of a representation of the group H that is indecomposable over K_P and decomposable over K'.

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Note: Figure translations are in progress. See original paper for figures.