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Abstract

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MATHEMATICS

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ON THE CONTINUOUS DEPENDENCE ON A PARAMETER OF THE FUNDAMENTAL SOLUTION OF A PARABOLIC SYSTEM

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1. Let E_n denote the n-dimensional Euclidean space of points $x = (x_1, \dots, x_n)$. Consider, in the $(n+1)$-dimensional space $(-\infty, +\infty) \times E_n$ $(-\infty < t < +\infty,\ x \in E_n)$, the system of differential equations written in matrix form

$$\frac{\partial u}{\partial t} - \sum_{r=1}^{2p} \sum_{i_1,\dots,i_r=1}^{n} A^{(i_1,\dots,i_r)}(t,x,\sigma) \frac{\partial^r u}{\partial x_{i_1} \cdots \partial x_{i_r}} - A(t,x,\sigma)u = 0, \qquad (B(\sigma))$$

where σ is a parameter varying in some neighborhood of the point $\sigma = 0$. We shall assume that the system

$$\frac{\partial u}{\partial t} - \sum_{r=1}^{2p} \sum_{i_1,\dots,i_r=1}^{n} A^{(i_1,\dots,i_r)}(t,x) \frac{\partial^r u}{\partial x_{i_1} \cdots \partial x_{i_r}} - A(t,x)u = 0, \qquad (B(0))$$

obtained from $B(\sigma)$ for $\sigma = 0$, is parabolic in the sense of I. G. Petrovskii ([1]). This means that, for any real $\alpha_1, \dots, \alpha_n$, the roots of its characteristic equation (E is the identity matrix)

$$\det\left[(-1)^p \sum_{i_1,\dots,i_{2p}=1}^{n} A^{(i_1,\dots,i_{2p})}(t,x)\alpha_{i_1} \cdots \alpha_{i_{2p}} - \lambda E\right] = 0$$

satisfy the inequality

$$\operatorname{Re}\lambda < -\delta\left(\sqrt{\alpha_1^2+\dots+\alpha_n^2}\right)^p \qquad (\delta>0).$$

In the present paper we investigate the behavior of the fundamental matrix solution $U(t,x;\tau,y;\sigma)$ of the system $B(\sigma)$ under small changes of the parameter σ.

Let the following conditions be satisfied:

1. Every coefficient $A^{(i_1,\dots,i_r)}(t,x)$ of the system $B(0)$ is continuous and satisfies the inequalities

$$\left|A^{(i_1,\dots,i_r)}(t,x)\right| \le M, \tag{1}$$

$$\left|A^{(i_1,\dots,i_r)}(t,x)-A^{(i_1,\dots,i_r)}(t,y)\right| \le M|x-y|^\gamma \qquad (M>0,\quad 0<\gamma\le 1).$$

2. The leading coefficients of the system are uniformly continuous in the variable t.

The second condition may be replaced by the requirement of strong ellipticity of the differential expression

$$\sum_{i_1,\dots,i_{2p}=1}^{n} A^{(i_1,\dots,i_{2p})}(t,x)\frac{\partial^{2p}}{\partial x_{i_1}\cdots\partial x_{i_{2p}}}.$$

It is known ([2,3]) that under the indicated conditions the system $B(0)$ has a fundamental solution $U(t,x;\tau,y)$, satisfying for $t>\tau$

inequality

$$\left|D_x^l U(t,x;\tau,y)\right| \le \frac{C}{(t-\tau)^{(n+l)/2p}}\exp\left\{a(t-\tau)-\eta\left[\frac{|x-y|}{\sqrt[2p]{t-\tau}}\right]^{2p/(2p-1)}\right\} \qquad (0\le l\le 2p), \tag{2}$$

where C, a, and η are positive constants depending only on M, γ, and the constant of parabolicity δ^*.

II. We now formulate two propositions on the continuous dependence, with respect to a parameter, of the fundamental solution of the system $B(\sigma)$.

Theorem 1. *Suppose that the following conditions are satisfied: a) for all values of σ from some neighborhood of the point $\sigma=0$, the coefficients of the system $B(\sigma)$ satisfy conditions 1 and 2 of item I with constants M and γ independent of σ; b) uniformly with respect to t and x,*

$$\lim_{\sigma\to 0} A^{(i_1,\dots,i_r)}(t,x,\sigma) = A^{(i_1,\dots,i_r)}(t,x). \tag{3}$$

Then for every $\varepsilon > 0$ there exists a neighborhood of the point $\sigma = 0$ in which, for $t > \tau$, the inequalities hold:

$$\left|D_x^l U(t,x;\tau,y;\sigma) - D_x^l U(t,x;\tau,y)\right| \le$$

$$\le \frac{C\varepsilon}{(t-\tau)^{(n+l)/2p}} \exp\left\{a(t-\tau) - \eta\left[\frac{|x-y|}{\sqrt[2p]{t-\tau}}\right]^{2p/(2p-1)}\right\} \quad (0 \le l \le 2p-1). \tag{4}$$

Theorem 2. *Suppose that the following requirements are satisfied: a) for any fixed value of σ from some neighborhood of $\sigma = 0$, the coefficients of the system $B(\sigma)$ are continuous in $(n+1)$-dimensional space, and its senior coefficients are uniformly continuous in t; b) uniformly with respect to t and x, the equalities (3) and*

$$\lim_{\sigma\to 0} \frac{\left|A^{(i_1,\dots,i_r)}(t,x,\sigma) - A^{(i_1,\dots,i_r)}(t,y,\sigma) - A^{(i_1,\dots,i_r)}(t,x) + A^{(i_1,\dots,i_r)}(t,y)\right|}{|x-y|^\gamma} = 0.$$

Then for every $\varepsilon > 0$ there exists a neighborhood of the point $\sigma = 0$ in which inequality (4) is also valid for $l = 2p$.

III. We give some applications of Theorem 1.

Theorem 3. *For any s from the segment $[\tau, t]$, the equality holds*

$$U(t,x;\tau,y) = \int_{E_n} U(t,x;s,z)\, U(s,z;\tau,y)\, dz.$$

Theorem 4. *The following equalities hold:*

$$\lim_{\tau\to t-0} \frac{1}{t-\tau}\left[\int_{E_n} U(t,x;\tau,y)\, dy - E\right] = A(t,x),$$

$$\lim_{\tau\to t-0} \frac{1}{t-\tau} \int_{E_n} \prod_{k=1}^{r} (y_{i_k} - x_{i_k})\, U(t,x;\tau,y)\, dy =$$

$$= \begin{cases} r!\, A^{(i_1,\dots,i_r)}(t,x), & \text{for } 1 \le r \le 2p, \\ 0, & \text{for } r > 2p. \end{cases}$$

* In (2) the parabolic system is considered only for t and τ varying in some finite segment $[0, T]$. Therefore, in the estimate obtained there, $a = 0$, and C also depends on T. In (3) the estimate (2) is given under the assumption that the coefficients of the system depend only on x.

Theorem 5. In order that, for all $t > \tau$, the equality

$$\int_{E_n} U(x, x; \tau, y)\, dy = E$$

hold, it is necessary and sufficient that $A(t, x) = 0$.

These three theorems were proved in paper [4] (see also [5]) under strong restrictions, by means of Green' s formula, estimates, and the properties of the fundamental matrix following from its definition. They can now be proved under the weak restrictions of § 1 by means of the following device.

For a given parabolic system $B(0)$ one constructs a system $B(\sigma)$, whose coefficients for $\sigma \neq 0$ are defined by the equalities

$$A^{(i_1,\dots,i_r)}(t, x, \sigma) = \int_{E_n} K(|\sigma|, x - y) A^{(i_1,\dots,i_r)}(t, y)\, dy,$$

where

$$K(t, x) = \frac{1}{(\sqrt{2\pi t})^n} e^{-|x|^2/4t}$$

is the fundamental solution of the heat equation.

It is easy to see that the matrices $A^{(i_1,\dots,i_r)}(t, x, \sigma)$ satisfy the conditions of Theorem 1. Moreover, for any $\sigma \neq 0$ they have bounded derivatives with respect to x of arbitrary order.

Applying now, as in [4], Green' s formula to the fundamental solution of the system $B(\sigma)$ (it will be parabolic for every σ) and then letting σ tend to zero, we obtain the relations from which the propositions formulated above follow.

In conclusion, we note that if the conditions of the first two theorems are satisfied for the derivatives of the coefficients with respect to x up to order $m > 0$, then the order of the derivatives for which inequality (4) is valid is correspondingly increased.

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Note: Figure translations are in progress. See original paper for figures.