Reports of the Academy of Sciences of the USSR

LIN TSZUN-CHI

1964

SovietRxiv

Abstract

Full Text

Reports of the Academy of Sciences of the USSR
1964. Vol. 157, No. 3

MATHEMATICS

LIN TSZUN-CHI

ASYMPTOTICS OF SOLUTIONS OF THE CAUCHY PROBLEM IN THE CASE WHERE THE LIMITING EQUATION HAS A SINGULARITY

(Presented by Academician I. G. Petrovskii, February 3, 1964)

1. Consider the ordinary differential equation

$$L_\varepsilon y \equiv \varepsilon^2 y'' + a(x)y' + b(x)y = f(x) \tag{1}$$

and the initial conditions:

$$y(x)\big|_{x=0} = c_0, \qquad y'(x)\big|_{x=0} = 0, \tag{2}$$

where ε is a small positive parameter, the coefficients $a(x)$, $b(x)$ are positive for $x > 0$, and the expansions

$$\begin{gathered} a(x) = x + a_2x^2 + a_3x^3 + \cdots, \\ b(x) = b_0 + b_1x + b_2x^2 + \cdots, \\ f(x) = f_0 + f_1x + f_2x^2 + \cdots \end{gathered}$$

hold.

For $\varepsilon = 0$ one obtains the limiting equation

$$L_0 y \equiv a(x)y' + b(x)y = f(x).$$

It has a singularity at the initial point $x = 0$.

The aim of the present note is to construct the asymptotics, with respect to the small parameter ε, of the solution of the Cauchy problem (1), (2). We shall consider in detail the case $b_0 = 1$. (The case of arbitrary $b_0 > 0$ is qualitatively analogous.) To construct the asymptotic expansion of the solution y_ε of problem (1), (2), we divide the domain into three zones: in the first zone the limiting equation has a singularity, while in the second and third zones it has no singularities; moreover, in each zone the differential equation is solved in a different way. In this note we construct the asymptotics of the solution of problem (1), (2) only in the first two zones; in the third zone the asymptotics is constructed in the same way as in the paper (^{1}), and we shall not repeat it.

2. Calculation of the solutions in the first zone. Let $x = \varepsilon t$; rewrite equation (1) and conditions (2) in the form

$$L_\varepsilon y \equiv M_0 y + \varepsilon M_1 y + \cdots + \varepsilon^i M_i y + \cdots = f_0 + \varepsilon t f_1 + \cdots + \varepsilon^i t^i f_i; \qquad (1^*)$$

$$y(t)\big|_{t=0} = c_0, \qquad y'(t)\big|_{t=0} = 0, \qquad (2^*)$$

where

$$M_0 y = y''_t + t y'_t + y, \qquad M_i y = a_{i+1} t^{i+1} y'_t + b_i t^i y, \qquad i = 1, 2, \ldots$$

We shall seek the solution y_ε of problem (1*), (2*) in the form

$$y_\varepsilon = y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots + \varepsilon^i y_i + \cdots \qquad (3)$$

Substituting this expansion into (1*), (2*) and equating coefficients of like powers of ε, we obtain

$$M_0 y_0 = f_0, \qquad y_0(t)\big|_{t=0} = c_0, \qquad y'_0(t)\big|_{t=0} = 0; \qquad (4)$$

$$M_0 y_i = t^i f_i - \sum_{\gamma=1}^{i} M_\gamma y_{i-\gamma}, \qquad y_i(t)\big|_{t=0} = 0, \qquad y'_i(t)\big|_{t=0} = 0, \quad i = 1, 2, 3, \ldots \qquad (5)$$

Solving problem (4), we obtain

$$y_0(t) = O(1) + O(1) e^{-t^2/2}, \qquad y'_0(t) = O(t) e^{-t^2/2}.$$

By the method of mathematical induction it is not difficult to show that the solution $y_i(t)$ and its derivative have the asymptotic estimates

$$\begin{aligned} y_i(t) &= O(P_i(t^{3i})) e^{-t^2/2} + O(R(t^i)), \\ y'_i(t) &= O(P_i(t^{3i+1})) e^{-t^2/2} + O(R(t^{i-1})), \end{aligned} \qquad (5^*)$$

where $P_i(t^{3i})$ and $R(t^i)$ are polynomials in t of degrees not higher than $3i$ and i, respectively.

We shall assume that the first zone ends at $x = x_k = \varepsilon^{1/2}$ $(t = t_k = \varepsilon^{-1/2})$. Then to this value $x = \varepsilon^{1/2}$ $(t = \varepsilon^{-1/2})$ there will correspond the values

$$y\big|_{x=\varepsilon^{1/2}} = O(1), \qquad y'(x)\big|_{x=\varepsilon^{1/2}} = \varepsilon^{-1} y'(t)\big|_{t=\varepsilon^{-1/2}} = O(1).$$

3. Computation of the solution in the second zone. Having passed through the first zone, the curve enters the second zone. In this zone we solve the Cauchy problem for equation (1) with the initial conditions

$$y(x)\big|_{x=\varepsilon^{1/2}} = \bar{y}_0, \qquad y'(x)\big|_{x=\varepsilon^{1/2}} = \bar{P}_0. \tag{6}$$

Let $z = x\varepsilon^{-1/2}$; equation (1) and condition (6) are rewritten in the form

$$L_\varepsilon y \equiv \varepsilon y''_{zz} + A(z) y'_z + B(z) y = F(z); \tag{7}$$

$$y(z)\big|_{z=1} = \bar{y}_0, \qquad y'_z(z)\big|_{z=1} = \bar{P}_0 \varepsilon^{1/2}, \tag{8}$$

where $A(z) = \varepsilon^{-1/2} a(z\varepsilon^{1/2})$; $\alpha z \leqslant A(z) \leqslant \beta(z)$; $\gamma \leqslant B(z) \leqslant \delta$; $F(z) \leqslant \bar{f}_0$; $\delta, \gamma, \beta, \alpha, \bar{f}_0$ are positive constants.

Along with the expansion (7) of the operator L_ε in powers of ε, we consider a second expansion of this operator near the point $z = 1$ $\left(\tau = \dfrac{z-1}{\varepsilon}\right)$:

$$\varepsilon L_\varepsilon y = M_0 y + \varepsilon M_1 y + \dots + \varepsilon^{iM}_{iy} + \cdots, \tag{9}$$

where

$$M_0 y = y''_{\tau\tau} + a_{10} y'_\tau, \qquad M_{iy} = \tau^{ia}_{1i} y'_\tau + \tau^{i-1} b_{1\,i-1} y, \qquad i = 1, 2, \dots, \qquad a_{10} > 0.$$

We shall seek the solution y_ε of problem (7), (8) in the form:

$$y_\varepsilon = \sum_{i=0}^{n} \varepsilon^{iy}_i + \sum_{i=0}^{n+1} \varepsilon^{i+1} v_i + R_n, \tag{10}$$

where the functions y_i are determined by the first iterative process, the functions of boundary-layer type v_i by the second, and R_n is the remainder term.

In the first iterative process, the approximate solution of equation (7) is sought in the form

$$\tilde{y}_\varepsilon = y_0 + \sum_{i=1}^{n} \varepsilon_i^{iy}. \tag{11}$$

In the second iterative process we shall seek the functions of boundary-layer type in the form

$$v = \varepsilon v_0 + \varepsilon \sum_{i=1}^{n+1} \varepsilon_i^{iv}. \tag{12}$$

Substituting (10), (11), (12), respectively, into (8), (7), (9) and comparing terms with like powers of ε, we obtain

$$A(z)y_0' + B(z)y_0 = F(z), \qquad y_0(z)\big|_{z=1} = \bar{y}_0; \tag{13}$$

$$A(z)y_i' + B(z)y_i = -y_{i-1}'', \qquad y_i(z)\big|_{z=1} = -v_{i-1}\big|_{\tau=0}, \qquad i = 1, 2, \dots; \tag{14}$$

$$M_0 v_0 = 0, \qquad v_0'\big|_{\tau=0} = -y_0'\big|_{z=1}. \tag{15}$$

$$M_0 v_i = -\sum_{\gamma=1}^{i} M_\gamma v_{i-\gamma}, \qquad v_i'\big|_{\tau=0} = y_i'\big|_{z=1}. \tag{16}$$

Solving problems (13), (15), we obtain

$$y_0(z) = O(1) + O(z^{-\gamma/\beta}), \qquad y_0''(z) = O(z^{-1}), \qquad v_0(\tau) = \frac{y_0'(1)}{a_{10}} e^{-a_{10}\tau}.$$

By the method of mathematical induction we have shown that the solution y_i of problem (14) has the asymptotic estimates

$$y_i(z) = O(z^{-i}) + O(z^{-\gamma/\beta}), \qquad y_{i-1}''(z) = O(z^{-i}) + O(z^{-\gamma/\beta-1}). \tag{17}$$

By the method of undetermined coefficients we found the solution v_i of problem (16) in the form

$$v_i(\tau) = P(\tau)e^{-a_{10}\tau}, \tag{18}$$

where $P(\tau)$ depends polynomially on τ.

Lemma. One can determine a number N, independent of ε, such that on the interval $[0, N]$ there exists a sign-preserving solution $s(x)$ of the equation conjugate to equation (1),

$$\varepsilon^2 s'' - (as)' + bs = 0.$$

Take two curves

$$\bar{u} = s_n + \varepsilon^{n+1} d, \qquad \underline{u} = s_n - \varepsilon^{n+1} d,$$

where d is some constant, and s_n is a partial sum of expansion (3) or (10) (see below).

By virtue of the lemma, using the inequality of S. A. Chaplygin (^{2}), we have proved that our solution is squeezed between these curves: $\underline{u} \leqslant y_\varepsilon \leqslant \bar{u}$; in particular, $-d\varepsilon^{n+1} \leqslant R_n = y_\varepsilon - s_n \leqslant d\varepsilon^{n+1}$, where d is a constant independent of ε.

Theorem 1. Every solution $y = y_\varepsilon(x)$ of the differential equation (1), satisfying the initial conditions (2) in the first zone: from $x = 0$ to $x = \varepsilon^{1/2}$ (from $t = 0$ to $t = \varepsilon^{-1/2}$), has the asymptotic representation

$$y_\varepsilon = s_{2n+1} + R_n = \sum_{i=0}^{2n+1} \varepsilon^i y_i + R_n,$$

where y_i are defined above (see (5*)); $|R_n| \leqslant d\varepsilon^{n+1}$ is the remainder term; d does not depend on ε.

Theorem 2. Every solution of the differential equation (7), satisfying the initial conditions (8) in the second zone, from $x = \varepsilon^{1/2}$ to $x = N$, has the asymptotic representation

$$y_\varepsilon = s_n + R_n = \sum_{i=0}^{n} \varepsilon^i y_i + \sum_{i=0}^{n+1} \varepsilon^{i+1} v_i + R_n,$$

where y_i and v_i are defined above (see (17) and (18)); the remainder term $|R_n| \leqslant d\varepsilon^{n+1}$. d does not depend on ε.

Remark 1. The solution of equation (1) can be constructed not only in the direction of positive values of x, but also in the direction of negative ones. One may consider the solution of our equation on the interval $[-N, N]$, inside which lies the point $x = 0$. In a neighborhood of this point a peculiar phenomenon of an internal boundary layer takes place.

Remark 2. Analogous results hold for equation (1) and the initial conditions

$$y(x)\big|_{x=0} = c_0, \qquad y'(x)\big|_{x=0} = \frac{c_1}{\varepsilon}.$$

Remark 3. All the results obtained by us extend without essential difficulties to a certain class of linear ordinary differential equations of higher orders, for example to the problem

$$L_\varepsilon y \equiv \varepsilon^2 y''' + x(1+O(x))y'' + (b_0 + O(x))y' +$$

$$+(c_0 + O(x))y = f(x),$$

$$y(x)\big|_{x=0} = \tilde{c}_0, \qquad y'(x)\big|_{x=0} = \frac{c_1}{\varepsilon}, \qquad y''(x)\big|_{x=0} = 0.$$

In conclusion I express my deep gratitude to my scientific adviser, Corresponding Member of the Academy of Sciences of the USSR L. A. Lyusternik, for his advice and systematic assistance in carrying out the present work.

Moscow State University
named after M. V. Lomonosov

Received
29 I 1964

REFERENCES

1. M. I. Vishik, L. A. Lyusternik, *UMN*, **12**, no. 5, 3 (1957).

2. S. A. Chaplygin, *A New Method of Approximate Integration of Differential Equations*, Moscow, 1950.

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.