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Abstract

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Yu. P. GINZBURG

ON THE FACTORIZATION OF ANALYTIC MATRIX-FUNCTIONS

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1. We shall consider matrix-functions $X(\xi)$, regular for $|\xi| < 1$, of finite order n, with determinant not identically equal to zero. We shall say that such a matrix-function belongs to the **class** $A^{(n)}$ if

$$\sup_{0 \le \rho < 1} \int_0^{2\pi} \ln^+ \|X(\rho e^{i\vartheta})\| \, d\vartheta < \infty,$$

to the **class** $D^{(n)}$, if the family of functions

$$I_\rho(\varphi) = \int_0^{\varphi} \ln^+ \|X(\rho e^{i\vartheta})\| \, d\vartheta \qquad (0 \le \rho < 1, \quad 0 \le \varphi \le 2\pi)$$

is uniformly absolutely continuous, to the **class** $H_\delta^{(n)}$ $(0 < \delta < \infty)$, if

$$\sup_{0 \le \rho < 1} \int_0^{2\pi} \|X(\rho e^{i\vartheta})\|^\delta \, d\vartheta < \infty,$$

to the **class** $B^{(n)}$, if $\sup_{|\xi|<1} \|X(\xi)\| < \infty$, and, finally, to the **class** $\widetilde{B}^{(n)}$, if $\|X(\xi)\| \le 1$ for $|\xi| < 1$.

It is not hard to see that

$$A^{(n)} \supset D^{(n)} \supset H_{\delta'}^{(n)} \supset H_{\delta''}^{(n)} \supset B^{(n)} \supset \widetilde{B}^{(n)} \qquad (\delta' < \delta''),$$

and that the matrix-function $X(\xi)$ belongs to the class $A^{(n)}$, $D^{(n)}$, $H_\delta^{(n)}$ $(\delta > 0)$, $B^{(n)}$ if and only if its elements belong to the corresponding scalar class ([1]).

In the present note we give "parametric representations" of the matrix classes $A^{(n)}$, $D^{(n)}$, $H_\delta^{(n)}$, $B^{(n)}$, $\widetilde{B}^{(n)}$, analogous to those known ([1,2]) for the scalar case. In proving them we make essential use of methods developed by V. P. Potapov in the works ([3,4]), and in particular rely on his fundamental result concerning matrix-functions of the class $\widetilde{B}^{(n)}$ (a particular case of the theorem concerning matrix-functions bounded in the indefinite metric ([4])):

A matrix-function $X(\xi)$ *belongs to the class* $\widetilde{B}^{(n)}$ *if and only if*

$$X(\xi) = \Pi(\xi) \int_0^l \exp\{k[\xi, \vartheta(t)]\} \, dE(t) \, U, \tag{1}$$

where $\Pi(\xi)$ *is a product of Blaschke type, computed by the formula*

$$\Pi(\xi) = \prod_{j=1}^{r} \left[\frac{\xi_j - \xi}{1 - \bar{\xi}_j \xi} \frac{|\xi_j|}{\xi_j} P_j + (I - P_j) \right] \tag{2}$$

$$(r \leq \infty, \quad |\xi_j| < 1, \quad P_j \text{ are orthoprojections});$$

$\vartheta(t)$ is a nondecreasing scalar function ($0 \leq \vartheta(t) \leq 2\pi$), $E(t)$ is a Hermitian increasing matrix-function ($\operatorname{sp} E(t) = t$), U is a constant unitary matrix,

$$k(\xi, \vartheta) = \frac{\xi + e^{i\vartheta}}{\xi - e^{i\vartheta}}.$$

2. The following theorem gives “parametric representations”of the matrix classes $A^{(n)}$, $D^{(n)}$, $H_\delta^{(n)}$, $B^{(n)}$, $\widetilde{B}^{(n)}$.

Theorem 1. The matrix-function $X(\zeta)$ belongs to the class $A^{(n)}$ if and only if the representation

$$X(\zeta) = \Pi(\zeta) \cdot \prod_{j=1}^{s} \left[\int_0^{l_j} \exp\{k(\zeta, \vartheta_j) \, dE_j(t)\} \right] \times$$

$$\times \int_0^{2\pi} \exp\{k(\zeta, \vartheta) \, dS(\vartheta)\} \, U \int_0^{2\pi} \exp\{k(\zeta, \vartheta) \, M(\vartheta) \, d\vartheta\}, \tag{3}$$

holds, where $\Pi(\zeta)$ is a product of the Blaschke type (2), $s \leqslant \infty$, $0 \leqslant \vartheta_j \leqslant 2\pi$, $E_j(t)$ and $S(\vartheta)$ are Hermitian matrix-functions of bounded variation, with $S(\vartheta)$ a continuous singular function, and $M(\vartheta)$ a Hermitian summable matrix-function.

The function $X(\zeta)$ belongs: 1) to the class $D^{(n)}$, 2) to the class $H_\delta^{(n)}$, 3) to the class $B^{(n)}$ if and only if $X(\zeta)$ admits representation (3) with parameters satisfying, in addition to those listed above, the following conditions:

1) $E_j(t)$ and $S(\vartheta)$ are Hermitian increasing matrix-functions;

2) $E_j(t)$ and $S(\vartheta)$ are Hermitian increasing and

$$\int_0^{2\pi} e^{-2\pi\delta\lambda(\vartheta)}\,d\vartheta < \infty,$$

where $\lambda(\vartheta)$ is the smallest eigenvalue of the matrix $M(\vartheta)$;

3) $E_j(t)$ and $S(\vartheta)$ are Hermitian increasing, and $\lambda(\vartheta)$ is a function bounded below (in particular, $M(\vartheta) \geqslant 0$ for the class $\widetilde{B}^{(n)}$).

We indicate the main stages of the proof of this theorem. Using the fact that every function $X(\zeta) \in \widetilde{B}^{(n)}$ can, inside the unit disk, be uniformly approximated by finite products of Blaschke type (see ([4]), p. 133), and also using representation (1), one can prove the validity of the following proposition:

Lemma. If $X_j(\zeta) \in \widetilde{B}^{(n)}$ $(j = 1, 2)$, then there exist such $Y_j(\zeta) \in \widetilde{B}^{(n)}$ that

$$X_1(\zeta)X_2(\zeta) = Y_2(\zeta)Y_1(\zeta)$$

and

$$\det X_1(\zeta) = \det Y_1(\zeta), \qquad \det X_2(\zeta) = \det Y_2(\zeta).$$

Relying on the lemma, as well as on the equality

$$\det \int_0^l \exp\{f(t)\,dG(t)\} = \exp\left\{\int_0^l f(t)\,d[\operatorname{sp} G(t)]\right\}, \tag{4}$$

one can show that

$$I(\zeta) \equiv \int_0^l \exp\{k[\zeta, \vartheta(t)]\,dE(t)\} =$$

$$= \prod_{j=1}^{s}\left[\int_0^{l_j} \exp\{k(\zeta, \vartheta_j)\,dE_j(t)\}\right] \cdot \int_0^{l'} \exp\{k[\zeta, \tilde{\vartheta}(t)]\,d\widetilde{E}(t)\}U. \tag{5}$$

Here U is a constant unitary matrix; $s(\leqslant \infty)$ is the number of intervals on which the function $\vartheta(t)$ is constant; ϑ_j is the value of $\vartheta(t)$ on the j-th such interval; l_j is the length of this interval; $E_j(t)$ and $\widetilde{E}(t)$ are Hermitian increasing matrix-functions; $\tilde{\vartheta}(t)$ is a strictly increasing scalar function. The latter circumstance makes it possible, in the multiplicative integral occurring on the right in (5), to make a change of variable and write it in the form

$$\int_0^{2\pi} \exp\{k(\zeta,\vartheta)\, d\Sigma(\vartheta)\},$$

where $\Sigma(\vartheta)$ is a continuous, nondecreasing Hermitian matrix function on $[0, 2\pi]$.* Representing $\Sigma(\vartheta)$ as the sum of a singular and an absolutely continuous function, and using the lemma, equality (4), and also certain known estimates (see (4), p. 229) for the multiplicative integral, we obtain a proof of the theorem for the class $\widetilde{B}^{(n)}$, and hence also for the class $B^{(n)}$. In order now to obtain a proof of representation (3) for the class $A^{(n)}$, it is enough to note that $X(\zeta) \in A^{(n)}$ if and only if $X(\zeta) = y^{-1}(\zeta)Y(\zeta)$, where $y(\zeta)$ is a scalar function of class $\widetilde{B}$, not vanishing for $|\zeta| < 1$, and $Y(\zeta) \in \widetilde{B}^{(n)}$.

From the theorem on the parametric representation of the scalar class D (1) it follows that, for $X(\zeta)$ to belong to the class $D^{(n)}$, it is necessary and sufficient that

$$X(\zeta) = y^{-1}(\zeta)Y(\zeta), \quad \text{where } Y(\zeta) \in \widetilde{B}^{(n)}, \quad y(\zeta) = \exp\left\{\int_0^{2\pi} k(\zeta,\vartheta)m(\vartheta)\, d\vartheta\right\}$$

($m(\vartheta)$ is a scalar function summable on $[0, 2\pi]$). Hence the validity of Theorem 1 for the class $D^{(n)}$ follows.

As for the proof of the theorem in the part relating to the classes $H_\delta^{(n)}$, it is carried out using the equality

$$\|X(e^{i\vartheta})\| = \exp\{-2\pi\lambda(\vartheta)\}, \tag{6}$$

valid almost everywhere on $[0, 2\pi]$ for functions $X(\zeta) \in A^{(n)}$ represented by formula (3). Equality (6), which is the source of criteria for functions from $D^{(n)}$ to belong to one or another narrower class, can be established with the help of Theorem 3 below and theorem (1) on the recovery of a maximal scalar function of class D from the moduli of its boundary values.**

3. A matrix function of class $D^{(n)}$ is called **inner** if its boundary values on the unit circle are unitary almost everywhere. A function $X(\zeta) \in D^{(n)}$ is called **outer** if, for every function $Y(\zeta) \in D^{(n)}$ satisfying almost everywhere on $[0, 2\pi]$ the relation

$$Y^*(e^{i\vartheta})Y(e^{i\vartheta}) = X^*(e^{i\vartheta})X(e^{i\vartheta}), \tag{7}$$

the inequality $Y^*(\zeta)Y(\zeta) \le X^*(\zeta)X(\zeta)$ holds for $|\zeta| < 1$. It is obvious that if (7) holds for two outer functions $X(\zeta)$ and $Y(\zeta)$ from $D^{(n)}$, then $Y(\zeta) = UX(\zeta)$ (U is a constant unitary matrix).

Theorem 2. *A matrix function $X(\zeta) \in D^{(n)}$ is inner (outer) if and only if, in its representation (3), $M(\vartheta) \equiv 0$ (respectively $\Pi(\zeta) \equiv 1$, $E_j(t) = \text{const}$, $S(\vartheta) = \text{const}$).*

The following proposition is used in one of its parts (together with the fact that boundedness of a function from D on the unit circle implies its boundedness for $|\zeta| < 1$) for the proof of Theorem 2, while in another part it is obtained as its consequence:

In order that $X(\zeta)\,(\in D^{(n)})$ be inner (outer), it is necessary and sufficient that its determinant $\det X(\zeta)$ be an inner (outer) scalar function.

The theorem below on the uniqueness of the multiplicative representation of an outer function generalizes and refines the corresponding result of L. A. Sakhnovich (8).

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* We note that P. Masani (5,6) made an error in changing variables in the integral $I(\zeta)$ with a function $\vartheta(t)$ which, generally speaking, has intervals of constancy, and as a consequence arrived at an incorrect result.

** In the case of boundedness of $M(\vartheta)$ on $[0, 2\pi]$, formula (6) follows directly from L. A. Sakhnovich's theorem (7) on the boundary values of the multiplicative integral.

Theorem 3. If

$$X_j(\xi) = \int_0^{2\pi} \exp\{k(\xi, \vartheta) M_j(\vartheta)\, d\vartheta\} \qquad (j = 1, 2;\ |\xi| < 1),$$

where $M_j(\vartheta)$ are Hermitian matrix functions summable on $[0, 2\pi]$, and

$$X_1^*(e^{i\vartheta}) X_1(e^{i\vartheta}) = X_2^*(e^{i\vartheta}) X_2(e^{i\vartheta})$$

almost everywhere on $[0, 2\pi]$, then $M_1(\vartheta) = M_2(\vartheta)$ almost everywhere on $[0, 2\pi]$.

For the proof, consider the matrix functions

$$X_j(\xi; t) = \int_t^{2\pi} \exp\{k(\xi, \vartheta) M_j(\vartheta)\, d\vartheta\} \qquad (j = 1, 2;\ t \in [0, 2\pi]).$$

It is not difficult to see that

$$X_1^*(e^{i\vartheta}; t) X_1(e^{i\vartheta}; t) = X_2^*(e^{i\vartheta}; t) X_2(e^{i\vartheta}; t)$$

for each t and almost all ϑ in $[0, 2\pi]$. Since, on the basis of Theorem 2, the functions $X_j(\xi; t) \in D^{(n)}$ are outer, it follows that

$$X_2(\xi; t) = U(t) X_1(\xi; t) \tag{8}$$

($U(t)$ is a unitary matrix for each $t \in [0, 2\pi]$). We now note that

$$X_j(0;t) = \int_t^{2\pi} \exp\{-M_j(\vartheta)\, d\vartheta\}, \qquad (M_j^*(\vartheta) = M_j(\vartheta)),$$

and, as V. P. Potapov showed (see ([4]), p. 173), in this case the family of matrices $M_j(t)^*$ is uniquely recovered from the family of moduli

$$R_j(t) = \sqrt{X_j^*(0;t)X_j(0;t)}.$$

Since it follows from (8) that $R_1(t) = R_2(t)$ ($t \in [0, 2\pi]$), the theorem is proved.

4. The exposition above, although formally not relying on the theorem on optimal factorization in the class $H_2^{(n)}$ of summable positive matrix functions, first proved by M. G. Krein and presented in the works of Yu. A. Rozanov ([9–11]), has points of contact with it. If this theorem is used, it is not difficult to show that:

A Hermitian positive matrix $F(\vartheta)$ almost everywhere on $[0, 2\pi]$ is the modulus of boundary values of functions of the classes $A^{(n)}$ and $D^{(n)}$ if and only if

$$\left| \int_0^{2\pi} \ln \|F(\vartheta)\|\, d\vartheta \right| < \infty, \tag{9}$$

$$\left| \int_0^{2\pi} \ln \|F^{-1}(\vartheta)\|\, d\vartheta \right| < \infty. \tag{10}$$

Analogous assertions may be formulated for the classes $H_\delta^{(n)}$ $(0 < \delta < \infty)$ and $B^{(n)}$, replacing condition (9), respectively, by the conditions

$$\int_0^{2\pi} \|F(\vartheta)\|^\delta\, d\vartheta < \infty, \qquad \sup_{0 \le \vartheta \le 2\pi} \|F(\vartheta)\| < \infty.$$

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Odessa Pedagogical Institute
named after K. D. Ushinsky

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REFERENCES

1. I. I. Privalov, *Boundary properties of analytic functions*, 1950.

2. K. Hoffman, *Banach spaces of analytic functions*, 1963.

3. V. P. Potapov, DAN, **72**, No. 5 (1950).

4. V. P. Potapov, Tr. Moscow Math. Soc., **4**, 125 (1955).

5. R. Masani, C. R., **249**, 906 (1959).

6. R. Masani, C. R., **251**, 318 (1960).

7. L. A. Sakhnovich, UMN, **12**, issue 3 (75), 205 (1957).

8. L. A. Sakhnovich, Tr. III All-Union Math. Congress, 1, 1956, p. 120.

9. Yu. A. Rozanov, Tr. V All-Union Conf. on Functional Analysis, Baku, 1961, p. 122.

10. Yu. A. Rozanov, UMN, **13**, issue 2 (80), 93 (1958).

11. Yu. A. Rozanov, *Stationary random processes*, 1963.

* In ([4]) the arguments are carried out for the case $JM_j(\vartheta) \geq 0$, but in the part of interest to us they rely essentially only on the J-Hermiticity of $M_j(\vartheta)$ $(JM_j(\vartheta) = M_j^*(\vartheta)J)$.

Note: Figure translations are in progress. See original paper for figures.