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Abstract

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MATHEMATICS

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ON DENSITY CONDITIONS FOR THE INITIAL MANIFOLD

FOR AN ABSTRACT CAUCHY PROBLEM

(Presented by Academician S. N. Bernstein on 20 XI 1963)

Consider the abstract Cauchy problem

$$\frac{dx(t)}{dt} = Ax(t) \quad (0 \leq t < \infty), \quad x(0) = x_0, \quad (1)$$

with a linear operator A in a Banach space. Denote by I_A the linear manifold of those initial vectors x_0 for which there exists a smooth (in the strong sense) solution of problem (1). We shall be interested in the question under what conditions the manifold I_A is dense. That this is not always the case is shown by the following simple theorem.

Theorem 1. *Let A be a symmetric operator in a Hilbert space and let $\operatorname{def} A > 0$ in the lower half-plane. Then the manifold I_{iA} is orthogonal to all eigenvectors of the operator A^* corresponding to eigenvalues $\lambda = \alpha + \beta i$ ($\beta > 0$).*

Proof. Let $x_0 \in I_{iA}$, and let $x(t)$ be the corresponding solution. Further, let $A^*u = \bar{\lambda}u$, $u \neq 0$, $\operatorname{Im} \lambda < 0$. Then

$$\frac{d}{dt}(x(t), u) = i(Ax(t), u) = i\lambda(x(t), u)$$

and, consequently,

$$(x(t), u) = (x_0, u)e^{i\lambda t} \quad (0 \leq t < \infty). \quad (2)$$

Since $\|x(t)\| = \operatorname{const}$, by the symmetry of the operator A , it follows from (2), as $t \rightarrow \infty$, that $(x_0, u) = 0$.

From this theorem and a theorem of M. G. Krein ⁽¹⁾ there follows

Corollary. *If the operator A , satisfying the conditions of Theorem 1, is maximal and simple, then $I_{iA} = 0$.*

Denote by Γ a curve in the plane of the complex variable λ , given by the equation $\lambda = \lambda(s)$ ($-\infty < s < \infty$), where $\lambda(s)$ is a piecewise smooth function satisfying the following conditions: 1) $\lambda'(s \pm 0) \neq 0$ ($-\infty < s < \infty$); 2) $\lambda(s_1) \neq \lambda(s_2)$ ($s_1 \neq s_2$); 3) $|\lambda(s)| \rightarrow \infty$ ($|s| \rightarrow \infty$); 4) $|\lambda(s)|$ is a strictly monotone function in neighborhoods of the values $s = \pm\infty$; 5)

$$\sup_{-\infty < s < \infty} \operatorname{Re} \lambda(s) < \infty.$$

Let α be such that the half-plane $\operatorname{Re} \lambda < \alpha$ contains the entire curve Γ . Denote by Π_- the connected component of the complement of Γ lying in the indicated half-plane. Denote the second component by Π_+ .

Our constructions are based on the following simple lemmas, the proofs of which we omit.

Lemma 1. *Let the operator A have no spectrum on the line Γ , and let a continuous scalar function $\varphi(\lambda)$ ($\lambda \in \Gamma$) be such that*

$$\int_{\Gamma} \|\lambda \varphi(\lambda) R_{\lambda}\| |d\lambda| < \infty. \quad (3)$$

where R_{λ} , as usual, is the resolvent of the operator A . If, in addition,

$$\int_{\Gamma} \varphi(\lambda) e^{\lambda t} d\lambda = 0 \quad (0 \leq t < \infty), \quad (4)$$

then the range of the operator

$$S_{\varphi} = \int_{\Gamma} \varphi(\lambda) R_{\lambda} d\lambda$$

is contained in the manifold I_A .

Lemma 2. Let $\varphi(\lambda)$ ($\lambda \in \Gamma$) be a summable* scalar function locally satisfying the Lipschitz-Hölder condition. In order that (4) hold, it is necessary that $\varphi(\lambda)$ admit an extension $\hat{\varphi}(\lambda)$ ($\lambda \in \bar{\Pi}_-$), analytic in Π_- and continuous in $\bar{\Pi}_-$. This is also sufficient under the additional condition

$$\lim_{|\lambda| \rightarrow \infty} \hat{\varphi}(\lambda) = 0 \quad (\lambda \in \Pi_-).$$

We shall now subject the curve Γ to one more restriction, namely, we shall require that for some $p > 1$

$$\int_{\Gamma} \frac{|d\lambda|}{1 + |\lambda|^p} < \infty.$$

Theorem 2. Let the operator A , with dense domain of definition, have no spectrum on the curve Γ and in the domain Π_+ , and let

$$\|R_{\lambda}\| = O(1 + |\lambda|^m) \quad (\lambda \in \overline{\Pi}_+)$$

with some $m \geq 0$. Then the manifold I_A is dense.

For the proof it is enough to put

$$\varphi(\lambda) = \frac{1}{2\pi i(\lambda - a)^{[m+p+2]}}.$$

In this case the operator S_{φ} is readily computed by the usual methods of the theory of residues and turns out to be equal to $R_a^{[m+p+2]}$. Consequently, its range coincides with the domain of definition of the operator $A^{[m+p+2]}$, and the latter is dense (see, for example, (2), p. 290).

Certain assumptions about the structure of the operator A make it possible to substantially weaken the growth restriction on the resolvent.

Theorem 3. Let the operator A , with dense domain of definition, have no spectrum on the line Γ and in the domain Π_+ , and suppose that

$$\int_{\Gamma} \frac{\ln_+ \|R_{\lambda}\|}{1 + \omega^2(\lambda)} |d\omega(\lambda)| < \infty, \quad (5)$$

where $\omega(\lambda)$ is a function conformally mapping the domain Π_- onto the upper half-plane $\text{Im } \omega > 0$. Suppose that the function $\omega(\lambda)$ on Γ locally satisfies the Lipschitz-Hölder condition, and that the inverse function $\lambda(\omega)$ has the same property on the real axis. If the operator $(R_{\lambda_0})^*$, for some $\lambda_0 \in \overline{\Pi}_+$, has no invariant subspaces $\mathfrak{L} \neq 0$ on which its spectrum is concentrated at zero, then the manifold I_A is dense.

We indicate the proof of Theorem 3.

Mapping the domain Π_- onto the upper half-plane and using the Poisson integral, one can, by virtue of (5), construct a function $\varphi(\lambda)$ ($\lambda \in \overline{\Pi}_-$), holomorphic in the domain Π_- and continuous in $\overline{\Pi}_-$, so that condition (3) is fulfilled. Moreover, one can arrange that the following condi-

* This property holds automatically when (3) is fulfilled, since it is not hard to prove that $\lim_{|\lambda| \rightarrow \infty} \|\lambda R_\lambda\| \geq 1$.

conditions: a) $\varphi(\lambda) \neq 0$ ($\lambda \in \Pi_-$); b) $\lim_{|\lambda| \rightarrow \infty} \varphi(\lambda) = 0$ ($\lambda \in \Pi_-$); c) the boundary values of the function $\varphi(\lambda)$ locally satisfy the Lipschitz-Hölder condition.

According to Lemmas 1 and 2, the range of the operator S_φ is contained in I_A . But all essential properties of the function $\varphi(\lambda)$ are preserved under multiplication by $e^{k\lambda}$ ($k \geq 0$). Therefore it remains to prove that the ranges of the operators $S(k) \equiv S_{\varphi_k}$ ($\varphi_k(\lambda) = \varphi(\lambda)e^{k\lambda}$) form a complete system.

Let \mathcal{L} be the subspace of those linear functionals f for which $f(S(k)x) = 0$ for all $k \geq 0$ and all x , i.e.

$$\int_{\Gamma} \varphi(\lambda) f(R_\lambda x) e^{k\lambda} d\lambda = 0.$$

By Lemma 2, the function $\varphi(\lambda)f(R_\lambda x)$, and hence also $f(R_\lambda x)$ for any x , is analytically continued into the domain Π_- . But $f(R_\lambda x)$ is regular also in $\bar{\Pi}_+$. Therefore $f(R_\lambda x)$ is an entire function. Thus $(R_\lambda)^* f$ turns out to be an entire function for any $f \in \mathcal{L}$. But then the subspace \mathcal{L} , if it is nonzero, turns out to be endowed with precisely those properties which are forbidden by the formulation of the theorem. Consequently, $\mathcal{L} = 0$.

Let us note some particular cases of Theorem 3. If Γ is the imaginary axis, then condition (5) takes the form

$$\int_{-\infty}^{\infty} \frac{\ln_+ \|R_{i\omega}\|}{1 + \omega^2} d\omega < \infty.$$

If Γ is the boundary of the angle

$$(1 - q)\pi \leq \arg \lambda \leq (1 + q)\pi \quad (0 < q \leq 1/2),$$

then condition (5) takes the form

$$\int_0^{\infty} \frac{\ln_+ \|R_{-\omega e^{-iq\pi}}\| + \ln_+ \|R_{-\omega e^{iq\pi}}\|}{1 + \omega^{1+1/2q}} d\omega < \infty.$$

Finally, if Γ is the boundary of the half-strip $|\operatorname{Im} \lambda| \leq h$, $\operatorname{Re} \lambda \leq 0$, then condition (5) takes the form

$$\int_0^{\infty} \ln_+ \|R_{-\tau \pm ih}\| e^{-\pi\tau/2h} d\tau < \infty.$$

This series of conditions shows that the restriction on the growth of the resolvent weakens as the spectrum approaches the negative half-axis. The general character of this observation is clear from the results of S. E. Varshavskii ⁽³⁾, concerning the asymptotic behavior of conformal mappings.

The following theorem is proved analogously to Theorem 3.

Theorem 4. *Suppose that all the conditions of Theorem 3 are satisfied, except the one concerning the operator $(R_\lambda)^*$. If, in addition, the spectrum of the operator A is nonempty, then $I_A \neq 0$.*

The condition that the spectrum be nonempty is essential, as is shown by the following example, "dual" to the example constructed in ⁽⁴⁾.

Let A be the operator in $\mathcal{L}_\rho^2(0, \infty)$ defined by the differentiation operation d/ds and the boundary condition

$$\xi(0) = 0. \quad (6)$$

Then $I_A = 0$, since if the function $\xi(s+t)$ ($s, t \geq 0$) satisfies condition (6) for all $t \geq 0$, then it is identically equal to zero.

Let us subject the weight $\rho(s)$ to the conditions: 1) $\rho(s_1 + s_2) \leq \rho(s_1)\rho(s_2)$ ($s_1, s_2 \geq 0$); 2) $\sqrt{\rho(s)} e^{ks} \in \mathcal{L}(0, \infty)$ for all $k \geq 0$. One may take, for example, $\rho(s) = e^{-s^2}$. Then the operator A will have no spectrum.

Incidentally, the resolvent of the operator A constructed is estimated in the following way:

$$\|R_\lambda\| \leq \int_0^\infty \sqrt{\rho(s)} e^{ks} ds \quad (k = \operatorname{Re} \lambda),$$

whence it is clear that the conditions of Theorem 2 cannot be weakened so that the growth of the resolvent is restricted only on the line Γ .

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References

- ¹ M. G. Krein, *Ukr. Mat. Zh.*, No. 2, 3 (1949).
- ² E. Hille, *Functional Analysis and Semigroups*, II, 1951.
- ³ S. E. Varshavskii, *Sbornik of Translations. Mathematics*, 2, 4, 67 (1958).
- ⁴ Yu. I. Lyubich, *DAN*, 130, No. 5, 969 (1960).

Note: Figure translations are in progress. See original paper for figures.

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