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Abstract

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APPLICATION OF THE FINITE-DIFFERENCE METHOD TO THE SOLUTION OF BOUNDARY-VALUE PROBLEMS ON RIEMANN SURFACES

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Let R be a finite Riemann surface of genus ρ. Such a surface, as is known, is realized in the form of a bounded many-sheeted domain. We cover this many-sheeted domain G by a mesh in such a way that the branch points lie at its nodes, the mesh being the same for all sheets. We approximate, by finite-difference constructions, the equation

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + c\frac{\partial U}{\partial x} + d\frac{\partial U}{\partial y} + gU = f, \tag{1}$$

where c, d, g, f are continuous functions of the independent variables x and y, defined on each of the sheets of the domain G; $g \leqslant 0$. The boundary condition has the form

$$\alpha\frac{\partial U}{\partial n} + \beta U = \psi \quad \text{on } \Gamma, \tag{2}$$

where $\alpha\beta \geqslant 0$, $\alpha^2 + \beta^2 = 0$; Γ, the boundary of the domain G, is assumed closed and piecewise smooth.

To approximate equation (1) at the branch points it is more convenient to write it in terms of z and $\bar{z}$. As a local parameter at a branch point of order n we choose

$$w = \sqrt[n]{z - z(p_0)}\,. \tag{3}$$

Everywhere except at branch points, we take z as the local parameter. Owing to the covariant character of the coefficients of the first derivatives and of the tensor character of the coefficients g and f, and by virtue of (3), applying the mean-value theorem ([7],[8]), we obtain at a branch point the finite-difference equation corresponding to (1) in the form

$$l[U_{ik}] \equiv 4nU_{ik} - \sum_{m=0}^{n-1} \left[U_{i-1,k}^{(m)} + U_{i+1,k}^{(m)} + U_{i,k-1}^{(m)} + U_{i,k+1}^{(m)}\right] = 0. \qquad (4)$$

The index m denotes the number of the sheet of the n-sheeted domain. We have taken the mesh to be square in a neighborhood of the branch points, although in G as a whole the case of a nonuniform mesh is possible because of the necessity of placing the branch points at its nodes.

The error of approximation of equations (1), (2) by finite-difference constructions is obtained with accuracy up to h^2 (h is the mesh step) at ordinary points; at a branch point, in terms of its local parameter, we have the same accuracy if the factor h^{n-1} is introduced under the radical in (3), and if, in addition, the mean-value theorem is used, the error is considerably reduced.

To find the values U_{ik} at the interior nodes of the domain G, we obtain a system of linear algebraic equations in which the number of equations is equal to the number of unknowns; a branch point is counted once for all the sheets that it “fastens together.” This system, consisting of equations for ordinary points of the form ([10]), p. 446 on each of the sheets and equations (4) for branch points, supplemented by Collatz equations for boundary nodes from ([11]),

symbolically write in the form

$$\sum_{j=1}^{N} a_{ij}U_j = f_i \qquad (i = 1, 2, \dots, N), \qquad (5)$$

where N is the number of interior points on all sheets of the domain G.

Theorem 1. *The system of equations (5) always has a solution, and moreover a unique one, if $\beta \neq 0$ in (2).*

The proof follows from the existence, for the homogeneous system (5) (f_{ik} and all values at boundary nodes are equal to zero), of only the trivial solution. The latter holds by virtue of the maximum principle ([10], p. 446), which is satisfied here. For $\beta \equiv 0$ a solvability condition is written down, similarly to ([9]); for example, when $f = 0$ this condition has the form

$$\sum_{\nu=1}^{l} U_\nu = 0, \qquad (6)$$

where U_ν are the normal differences ([9]) on Γ; l is the number of nodal points on the boundary. For uniqueness, the value of U is prescribed at one of the points. The case $a \neq 0$ is considered under the condition that the branch points do not lie on the boundary of the domain. We solve system (5) by the iteration method. The presence of branch points and the passage from sheet to sheet (when some of the neighboring points lie on another sheet) do not impair convergence.

We prove convergence of the iterative process by using matrix calculus. The N unknowns entering system (5) are denoted as the N components of a certain vector ξ (a column matrix), the value of the ν-th approximation as the components of the vector $\xi^{(\nu)}$, and the free terms as the components of the vector η; then (5) takes the form $\xi^{(\nu+1)} = A\xi^{(\nu)} + \eta$ $(\nu = 0, 1, 2, ...)$.

The matrix A has the following properties: 1) on the main diagonal there are only zeros; 2) all elements are nonnegative; 3) the sum of all elements of each row does not exceed one; 4) the sum of all elements of at least one row is less than one.

The matrix A is positively definite; hence all eigenvalues λ of the matrix A (the roots of the equation $\det|A - \lambda E| = 0$, E being the identity matrix) are less than one; this is sufficient for convergence of the iterative process ([11]).

Theorem 2. *The sequences constructed by the iteration method as applied to (5) converge for arbitrary initial values.*

We now determine how close the values obtained from (5) are to the values of the exact solution of problem (1), (2) at the corresponding points. At ordinary points we obtain the inequalities indicated in ([4]), p. 281, while at branch points

$$|U - U_{ik}| \le \mu + \beta,$$

where $\beta = h^2 n M_4/24$; $\mu = \beta + 3h^2 M_2$; $U \in C^4$. Here M_2, M_4 are estimates of the moduli of the derivatives of the second and fourth orders, respectively; n is the order of the branch point.

A generalization of the preceding results to equations of the form

$$aU_{xx} + bU_{yy} + cU_x + dU_y + gU = f(x, y)$$

with variable coefficients under the condition $a > 0,\ b > 0,\ g \le 0$ was carried out for a surface of genus 1 with a hole, realized in the form of a parallelogram with pairwise identified sides and with a cut. For this case all results are obtained in the same degree of generality as for plane domains. Essential is the fact that, under the identifications, we lose as many equations as unknowns.

In the case of the Neumann problem, in order that U be single-valued, it is necessary and sufficient that the periods of U along the sides of the rectangle realizing the torus, and along Γ, be equal to zero. Taking (6) into account, this can be achieved ([1], p. 402).

The method of finite differences is, as is well known, not only a means of computing approximate solutions, but also a method for proving existence theorems. The proof is based on the compactness of the family of solutions of finite-difference equations. This compactness makes it possible to pass to the limit as $h \to 0$. Following ([2]), the existence of a solution of the Dirichlet problem for the Laplace equation on a torus with a hole has been proved. The uniform boundedness of U_h follows from the maximum principle, while the uniform continuity of U_h and of their difference quotients up to and including the second order is proved by introducing an auxiliary function as in ([2]) and taking into account the fact of identifying the sides of the rectangle realizing the torus. The inequalities obtained in the course of all the preceding arguments, as well as the stability of the computational methods, make it possible to judge the stability of the difference equations for the cases considered by us ([12]). At the Computing Center of Perm State University, on the "Aragats" computer, the Dirichlet problem on a surface of genus 10 was solved in 1.5 minutes; in 5 minutes the function $f(z) = U + iV$ was reconstructed on a torus with a curvilinear cut; solutions of the Neumann boundary-value problems and of the third boundary-value problem were obtained; the harmonic and quasiharmonic measure was found on a 30-sheeted domain and on a surface of genus 42, on a nonorientable surface (the Möbius strip). The computer programs are written so that the number of branch points, their location, and the number of sheets of the many-sheeted domain G can be varied. The method of matrix factorization ([5]) was also used for the computation; its stability is justified in ([6]). The methods of ([3]) make it possible to solve problems also for an infinite domain under certain restrictions.

The problem of transferring the method of nets to Riemann surfaces was posed by L. A. Lyusternik more than 30 years ago.

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named after A. M. Gorky

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CITED LITERATURE

[1] R. Nevanlinna, *Uniformization*, IL, 1955.
[2] I. G. Petrovskii, UMN, vol. 8 (1940).
[3] L. A. Lyusternik, UMN, vol. 8 (1940).
[4] L. V. Kantorovich, V. I. Krylov, *Approximate Methods of Higher Analysis*, M.–L., 1962.
[5] G. I. Marchuk, *Numerical Calculations of Nuclear Reactors*, 1961.
[6] V. V. Rusanov, *Computational Mathematics*, No. 6 (1960).
[7] S. Ya. Gusman, G. D. Merzlyakova, *Proceedings of the Perm Higher Command-Engineering School*, vol. 1 (1963).
[8] G. D. Merzlyakova, *Scientific Notes of Perm State University, Mathematics Series* (1963).
[9] D. A. Panov, UMN, 4 (1938).
[10] I. S. Berezin, N. P. Zhidkov, *Methods of Computation*, 2, 1962.
[11] L. Kollatz, *Numerical Methods for Solving Differential Equations*, 1953.

Note: Figure translations are in progress. See original paper for figures.