MATHEMATICS

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Abstract

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SOME PROPERTIES OF LINEAR SUPERPOSITIONS OF SMOOTH FUNCTIONS

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§ 1. **The "dimension" of a space of functions of n real variables.** Let G_n be a domain of n-dimensional Euclidean space; $C(G_n)$ the space of all continuous real functions in G_n. Functions $f_1(x), f_2(x) \in C(G_n)$ will be called (ε, δ)-distinguishable if there exists an n-dimensional ball $S_\delta \subset G_n$ of radius δ such that

$$\min_{x \in S_\delta} |f_1(x) - f_2(x)| \geq \varepsilon.$$

Denote by $N_{\varepsilon,\delta}(E)$ the number of elements of a maximal subset of $E \subset C(G_n)$, any two elements of which are (ε, δ)-distinguishable.

Theorem 1. Let a set $F \subset C(G_n)$, everywhere dense in the uniform metric, together with each element f, contain the element λf; $\rho(f)$ is a nonnegative functional, defined on F, such that for every $f \in F$ $\rho(\lambda f) \to 0$ as $\lambda \to 0$; $F_{\varepsilon,r}$ is the intersection of two sets

$$\{f : \max_{x \in G} |f(x)| \leq \varepsilon;\ f(x) \in C(G_n)\} \quad \text{and} \quad \{f : \rho(f) \leq r;\ f \in F\}.$$

Then, for every $r > 0$,

$$\lim_{\delta \to 0} \lim_{\varepsilon \to 0} \left[-\frac{\log \log N_{\varepsilon,\delta}(F_{\varepsilon,r})}{\log \delta} \right] = n.$$

From Theorem 1 we obtain that, for example, the space of analytic functions of n real variables and the space of all continuous functions of n real variables have the same dimension, equal to n. Relying on this property of the space of functions of n variables, we shall prove, for example, the existence of analytic functions of n variables not representable by superpositions of the form

$$\sum_{k=1}^{\nu} p_k f_k(q_k^1, q_k^2, \dots, q_k^{n-1}),$$

where $\{f_k\}$ are arbitrary continuous functions of $n-1$ variables, and $\{q_k^i = q_k^i(x_1, x_2, \dots, x_n)\}$ are arbitrary previously fixed continuously differentiable functions of n variables; p_k are fixed continuous functions of n variables. For simplicity of exposition we shall restrict ourselves to the consideration of functions of only one and two variables.

§ 2. **Level lines of smooth functions.**

Notation. z is an arbitrary point of the two-dimensional plane with coordinates x, y; $\operatorname{grad}[f(z)]$ is the gradient of the function $f(z)$; $e(f, t)$ is the level set t of the function $f = f(z)$; $\vec{\tau}(e, z)$ is the unit tangent vector to the line e at the point $z \in e$; $\gamma(\vec{\tau}_1, \vec{\tau}_2)$ is the absolute value of the acute angle between the vectors $\vec{\tau}_1, \vec{\tau}_2$; $d_1(e)$ is the one-dimensional diameter of the set e; $h_1(e)$ is the length of the set e; $h_2(e)$ is the area of the set e; $S(\delta, z)$ is the circle of radius δ with center at the point z; c_k $(k = 1, 2, ...)$ are constants.

Let $p = p(z)$ and $q = q(z)$ be functions defined in a simply connected closed domain G and possessing the following properties: a) $p(z)$ is continuous in G and has modulus of continuity $\omega(\delta)$, and $|p(z)| \leq M < \infty$; b) $\partial q/\partial x$ and $\partial q/\partial y$ have in the domain G the same modulus of continuity $\omega(\delta)$, and $0 < m' \leq$

$$\leqslant |\operatorname{grad}[q(z)]| \leqslant M' < \infty.$$

The constants $\{c_k\}$ will henceforth be assumed to depend only on $M, m', M', d_1(G), \omega(d_1(G))$.

Lemma 1. Let $z_1 \in e_1 = e(q_1, t_1)$ and $z_2 \in e_2 = e(q_1, t_2)$. Then

$$\gamma(\vec{\tau}(e_1, z_1), \vec{\tau}(e_2, z_2)) \leqslant c_1\omega(\delta), \qquad \text{where } \delta = d_1(z_1 \cup z_2).$$

Lemma 2. If $\beta(e) \leqslant \pi/4$ is the magnitude of the angle swept out by the tangent vector to the connected line e, then

$$h_1(e) \leqslant d_1(e)(1 + c_2\beta(e)).$$

Lemma 3. Let $q_1 = q_1(z)$ and $q_2 = q_2(z)$ be functions defined in the domain G, possessing property b), and such that for every $z \in G$

$$\gamma\{\vec{\tau}[e(q_2, q_2(z)), z], \vec{\tau}[e(q_1, q_1(z)), z]\} \geqslant \gamma_0 > 0 \qquad (\gamma_0 = \text{const}).$$

Let e'_{q_2} and e''_{q_2} be two level lines of the function q_2, and e'_{q_1} and e''_{q_1} level lines of the function q_1; let $[a', a'']$ be a segment of the line e'_{q_1} with endpoints $a' \in e'_{q_2}$ and $a'' \in e''_{q_2}$; let $[b', b'']$ be a segment of the line e''_{q_1} with endpoints $b' \in e'_{q_2}$ and $b'' \in e''_{q_2}$.

Then

$$h_1([b', b'']) \leqslant h_1([a', a''])(1 + c_3\omega(\delta)/\gamma_0),$$

where

$$\delta = d_1([a', a''] \cup [b', b'']).$$

Proof. Since

$$q_2(a'') - q_2(a') = q_2(b'') - q_2(b'),$$

we have

$$\int_{[a',a'']} \frac{\partial q_2}{\partial s}\, ds = \int_{[b',b'']} \frac{\partial q_2}{\partial s}\, ds.$$

Consequently,

$$\frac{\partial q_2(a^*)}{\partial s}\, h_1([a',a'']) = \frac{\partial q_2(b^*)}{\partial s}\, h_1([b_1',b_2'']),$$

where $\frac{\partial q_2(a^*)}{\partial s}$ and $\frac{\partial q_2(b^*)}{\partial s}$ are the derivatives at the points $a^* \in [a',a'']$ and $b^* \in [b',b'']$ along the lines $[a',a'']$ and $[b',b'']$, respectively. Denote by q_2^* the derivative of q_2 at the point b^* along the direction $\vec{\tau}(e'_{q_1},a^*)$, and put

$$\alpha = \gamma\{\vec{\tau}[e''_{q_1},b^*], \vec{\tau}[e'_{q_1},a^*]\}.$$

By Lemma 1,

$$\alpha \leqslant c_1\omega(\delta),$$

where

$$\delta = d_1([a',a''] \cup [b',b'']),$$

and therefore

$$\frac{\partial q_2(a^*)}{\partial s} = q_2^* + O(1)\omega(\delta) = \frac{\partial q_2(b^*)}{\partial s} + O(1)\left\{\left|q_2^* - \frac{\partial q_2(b^*)}{\partial s}\right| + \omega(\delta)\right\} =$$

$$= \frac{\partial q_2(b^*)}{\partial s} + O(1)\{\alpha + \omega(\delta)\} = \frac{\partial q_2(b^*)}{\partial s} + O(1)\omega(\delta).$$

Consequently,

$$h_1([b',b'']) = h_1([a',a''])\frac{\partial q_2(a^*)}{\partial s}\left(\frac{\partial q_2(b^*)}{\partial s}\right)^{-1} = h_1([a',a''])\times$$

$$\times\left(1 + O(1)\omega(\delta)\left(\frac{\partial q_2(b^*)}{\partial s}\right)^{-1}\right) = h_1([a',a''])\left(1 + O(1)\frac{\omega(\delta)}{\gamma_0}\right),$$

since

$$\left|\frac{\partial q_2(b^*)}{\partial s}\right| \geqslant |\operatorname{grad}[q_2(b^*)]| \sin\gamma_0.$$

The lemma is proved.

Lemma 4. Let $S(\delta,z) \subset G$; let $\mu_q(t)$ be the function equal to

$$\left[\delta^2 - (t - q(z))^2 |\operatorname{grad}[q(z)]|^{-2}\right]^{1/2}$$

on the interval

$$q(z) - \delta|\operatorname{grad}[q(z)]| \leqslant t \leqslant q(z) + \delta|\operatorname{grad}[q(z)]|$$

and equal to zero outside this interval. Then

$$\int_{-\infty}^{\infty} \left|\mu_q(t) - h_1(e(q,t)\cap S(\delta,z))\right| \, dt \leqslant c_4\omega(\delta)\delta^2.$$

Proof. Let $[a,b] \subset e(q,t)\cap S(\delta,z)$ be a segment of a level line with endpoints a and b lying on the boundary of $S(\delta,z)$; let

$$\alpha_1 = \gamma((\overline{za}, \operatorname{grad}[q(z)])), \qquad \alpha_2 = \gamma((\overline{zb}, \operatorname{grad}[q(z)])).$$

We have

$$|t-q(z)| = |q(a)-q(z)| = \left|\int_{[z,a]} \frac{\partial q}{\partial s}\, ds\right| =$$

$$= \delta\cos\alpha_1 \,|\operatorname{grad}[q(z)]|\,(1+O(1)\omega(\delta)).$$

Consequently,

$$\delta\sin\alpha_1 = \left[\delta^2 - (t-q(z)+O(1)\delta\omega(\delta))^2 |\operatorname{grad}[q(z)]|^{-2}\right]^{1/2}.$$

Similarly,

$$\delta\sin\alpha_2 = \left[\delta^2 - (t-q(z)+O(1)\delta\omega(\delta))^2 |\operatorname{grad}[q(z)]|^{-2}\right]^{1/2}.$$

If $\alpha_1 \geq c_5\omega(\delta)$ (c_5 is a sufficiently large constant), then $[a,b] = e(q,t)\cap S(\delta,z)$. By Lemmas 1 and 2,

$$h_1[a,b] = \delta(\sin\alpha_1 + \sin\alpha_2)(1+O(1)\omega(\delta)) =$$

$$= 2\{\delta^2 - (t-q(z)+O(1)\delta\omega(\delta))^2|\operatorname{grad}[q(z)]|^{-2}\}^{1/2} + O(1)\delta\omega(\delta),$$

and since for every t

$$h_1(e(q,t)\cap S(\delta,z)) \leq c_6\delta[1+\omega(\delta)],$$

we have

$$\int_{-\infty}^{\infty} \left|h_1(e(q,t)\cap S(\delta,z)) - \mu_q(t)\right| \, dt =$$

$$= \int_{q(z)-\theta}^{q(z)+\theta} \left|h_1(e(q,t)\cap S(\delta,z)) - \mu_q(t)\right| \, dt + O(1)\delta^2\omega(\delta),$$

where $\theta = \delta\cos[c_5\omega(\delta)]|\operatorname{grad}[q(z)]|$. Further,

$$\int_{q(z)-\theta}^{q(z)+\theta} \left|h_1(e(q,t)\cap S(\delta,z)) - \mu_q(t)\right| dt = O(1)\delta^2\omega(\delta)\int_{-1}^{1}\frac{dt}{\sqrt{1-t^2}},$$

since

$$h_1(e(q,t)\cap S(\delta,z)) = h_1([a,b]) = 2\{\delta^2 - (t-q(z)+O(1)\delta\omega(\delta))^2|\operatorname{grad}[q(z)]|^{-2}\}^{1/2} + O(1)\delta\omega(\delta).$$

The lemma is proved.

Lemma 5. Let $p(z), q(z)$ satisfy conditions a), b); $S(\delta,z)\subset G$; and let $f(t)$ be an arbitrary continuous function uniformly bounded in absolute value by a constant m. Then

$$\iint_{S(\delta,z)} p(u,v)f(q(u,v))\,du\,dv = p(z)|\operatorname{grad}[q(z)]|^{-1}\int_{-\infty}^{\infty} f(t)\mu_q(t)\,dt + \lambda(z)m\delta^2\omega(\delta),$$

where $|\lambda(z)|\le c_7$.

Proof.

$$\iint_{S(\delta,z)} p(u,v)f(q(u,v))\,du\,dv = p(z)\iint_{S(\delta,z)} f(q(u,v))\,du\,dv +$$

$$+O(1)m\delta^2\omega(\delta) = p(z)\int_{-\infty}^{\infty}\left\{f(t)\int_{e(q,t)\cap S(\delta,z)}|\operatorname{grad}[q(u,v)]|^{-1}\sqrt{(du)^2+(dv)^2}\right\}dt +$$

$$+O(1)m\delta^2\omega(\delta) = p(z)|\operatorname{grad}[q(z)]|^{-1}\times$$

$$\times\int_{-\infty}^{\infty}\left\{f(t)\int_{e(q,t)\cap S(\delta,z)}\sqrt{(du)^2+(dv)^2}\right\}dt + O(1)m\delta^2\omega(\delta) =$$

$$= p(z)|\operatorname{grad}[q(z)]|^{-1}\int_{-\infty}^{\infty} f(t)h_1(e(q,t)\cap S(\delta,z))\,dt + O(1)m\delta^2\omega(\delta) =$$

$$= p(z)|\operatorname{grad}[q(z)]|^{-1}\int_{-\infty}^{\infty} f(t)\mu_q(t)\,dt + O(1)m\delta^2\omega(\delta)$$

(see Lemma 4). The lemma is proved.

Lemma 6. Let a number $\alpha > 0$ and functions $p(z), q(z), f(t)$, satisfying the conditions of Lemma 5, be given. Then, if for every integer

$$k\left\{\min_{z\in G} q(z) \le t_k = k\delta\frac{\alpha}{m} \le \max_{z\in G} q(z)\right\}$$

and every integer

$$l\left\{\min_{z\in G} |\operatorname{grad}[q(z)]| \le t_l' = l\frac{\alpha}{m} \le \max_{z\in G} |\operatorname{grad}[q(z)]|\right\}$$

the inequality

$$\left|\int_{t_k-t_l'\delta}^{t_k+t_l'\delta} f(t)\sqrt{\delta^2-\left(\frac{t-t_k}{t_l'}\right)^2}\,dt\right| \le \alpha\delta^2,$$

then for every disk $S(\delta,z)\in G$

$$\left|\iint_{S(\delta,z)} p(u,v)f(q(u,v))\,du\,dv\right| \le c_8(a\delta^2+m\delta^2\omega(\delta)).$$

The proof is easily obtained from Lemma 5. Denote by $F = F(m, p_1, p_2, \dots, p_n, q_1, q_2, \dots, q_n)$ the set of functions of the form $f(z) = \sum_{i=1}^n p_i(z)f_i(q_i(z))$, where $\{p_i\}, \{q_i\}$ are fixed functions satisfying conditions a), b) (the constants M, m', M' do not depend on i), and the $f_i(t)$ are arbitrary measurable functions uniformly bounded in modulus by the constant m.

Put

$$R(f(z),\delta) = \max_{S(\delta,z)\in G}\left|\frac{1}{\pi\delta^2}\iint_{S(\delta,z)} f(u,v)\,du\,dv\right|.$$

Denote by $\mathcal{H}_\varepsilon^\delta(F)$ the ε-entropy of the space F, taking as the distance between functions $f_1(z), f_2(z)\in F$ the number $R(f_1(z)-f_2(z),\delta)$.

Theorem 2. *There exist constants A and B such that, if $\varepsilon \ge Am\omega(\delta)$, then*

$$\mathcal{H}_\varepsilon^\delta(F) \le \frac{B}{\delta}\left(\frac{m}{\varepsilon}\right)^2,$$

where A and B do not depend on m, ε, δ.

Proof. Without loss of generality, we may assume that the functions $\{f_i(t)\}$ are continuous and are equal to zero outside the intervals $\{[a_i, b_i]\}$, where $a_i = \min_{z\in G} q_i(z)$, $b_i = \max_{z\in G} q_i(z)$. In order to compute

$$f_\delta(z) = \frac{1}{\pi\delta^2} \iint_{S(\delta,z)\in G} f(u,v)\,du\,dv$$

to accuracy ε, it is sufficient to specify the values

$$v_i(t_k, t_l') = \frac{1}{\pi\delta^2} \int_{t_k - t_l'\delta}^{t_k + t_l'\delta} f_i(t) \sqrt{\delta^2 - \left(\frac{t - t_k}{t_l'}\right)^2}\, dt$$

to accuracy $\alpha = \pi\varepsilon/2nc_8$, and to assume δ so small that $\varepsilon \geq 2nc_8 m\omega(\delta)/\pi = Am\omega(\delta)$ (see Lemma 6). To record the numbers $v_i(t_k, t_l')$ (i, l fixed) to accuracy α, it is sufficient to have $N_{i,l} = c_9[\log m/\alpha + (b_i - a_i)m/\delta\alpha]$ binary digits. Here it should be taken into account that $v_i(t_k, t_l')$ and $v_i(t_{k+1}, t_l')$ are sufficiently close; therefore, for storing $v_i(t_{k+1}, t_l') - v_i(t_k, t_l')$ it is sufficient to have a number of digits not depending on m/α. Consequently, the total number of digits sufficient for recording the function $f_\delta(z)$ is

$$N = \sum_{i,l} N_{i,l} \leq B(n)c_{10} \left(\log \frac{m}{\alpha} + \frac{m}{\delta\alpha}\right) \frac{m}{\alpha}(M' - m') \leq \frac{B}{\delta} \left(\frac{m}{\varepsilon}\right)^2.$$

Consequently,

$$\mathcal{H}_\varepsilon^\delta(F) \leq \frac{B}{\delta} \left(\frac{m}{\varepsilon}\right)^2.$$

The theorem is proved.

Corollary. *If the functions $p_1, p_2, \dots, p_n$ continuous in the whole plane, the functions $q_1, q_2, \dots, q_n$ continuously differentiable, and a number $M > 0$ are fixed, then for every closed region G the set of superpositions of the form*

$$\sum_{k=1}^{n} p_k f_k(q_k),$$

where $\{f_k\}$ are arbitrary continuous functions uniformly bounded by the constant M, is nowhere dense in the space of all functions continuous on G.

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Note: Figure translations are in progress. See original paper for figures.