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Abstract

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A Generalized Analogy Between Mass-Transfer Coefficients in a Laminar Multicomponent Boundary Layer with an Arbitrary Pressure Gradient

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In a multicomponent boundary layer, because of the difference between the binary diffusion coefficients and possible homogeneous reactions, there is no similarity between the concentration fields of the individual components; nor is there similarity between the concentration and enthalpy fields. Consequently, in the general case there is no analogy between the mass-transfer coefficients, on the one hand, and the mass-transfer and heat-transfer coefficients, on the other.

However, as shown in papers (^{1,2}), in a frozen multicomponent boundary layer in the vicinity of the critical point (line) of a blunt body, by introducing the concepts of effective diffusion coefficients it is possible to establish a generalized analogy between the mass-transfer coefficients in the presence of moderate blowing, as well as a similarity of the concentration fields of components with close (equal) diffusion properties and disappearing simultaneously either at the wall or at the outer boundary of the boundary layer. The application of this analogy, together with the determination of the effective diffusion coefficients (^{1}), gives a system of algebraic equations for calculating the effective diffusion coefficients at the wall and the ratios of all mass-transfer coefficients to one another. As follows from its derivation, the generalized analogy remains valid for all self-similar solutions of the equations of a multicomponent boundary layer in the absence of homogeneous reactions.

In the present note this analogy between mass-transfer coefficients is generalized to the case of a multicomponent boundary layer with an arbitrary longitudinal pressure gradient and an arbitrary distribution of blowing along the surface. Using the concepts of effective diffusion coefficients introduced in (^{1}), the diffusion equations of the components are written in the usual form, but with their own variable diffusion coefficient. These equations are solved by the method of asymptotic integration for $S_i \to \infty$ and, for moderate blowing, by the method that was previously applied in (^{3}) to integrate the momentum equation for an incompressible boundary layer on an impermeable wall. As a result it is shown that the relations between the mass-transfer coefficients retain the structure established for self-similar solutions; however, the quantities entering them now depend on the longitudinal coordinate. It is interesting to note that the dependence of the indicated relations on the body shape is weak. Near the separation point the relations obtained are unsuitable and a special analysis is required.

1. With the aid of the effective diffusion coefficients D_i, defined from the expressions (1)

$$\frac{1}{D_i} = \sum_{j=1}^{N} \frac{x_j}{D_{ij}} \left(1 - \frac{c_i J_j}{c_j J_i}\right) + \sum_{k=1}^{N} c_k \sum_{j=1}^{N} \frac{x_j}{D_{kj}} \left(\frac{c_i J_j}{c_j J_i} - \frac{c_i J_k}{c_k J_i}\right) \qquad (i = 1, \dots, N), \tag{1}$$

the diffusive mass flux of the i-th component normal to the surface can be written in the form of Fick' s law

$$J_i = -\rho D_i \, dc_i/dy \qquad (i = 1, \dots, N). \tag{2}$$

Then the diffusion equation for component i assumes the usual form

$$\rho \left(u \frac{\partial c_i}{\partial x} + v \frac{\partial c_i}{\partial y}\right) = \frac{\partial}{\partial y} \left(\rho D_i \frac{\partial c_i}{\partial y}\right) \qquad (i = 1, \dots, N), \tag{3}$$

where x_i, c_i are the molar and mass concentrations of the i-th component; D_{ij} is the binary diffusion coefficient for the pair (i, j); ρ is the density of the mixture; u, v are the projections of the velocity vector on the tangent (the x-axis) and on the normal (the y-axis) to the surface; N is the number of components in the mixture. Adding to equations (3) the momentum and energy equations and transforming the resulting system to the Dorodnitsyn–Stepanov variables in the Lees form by the formulas

$$\xi = \int_0^x \mu_0 \rho_0 u_e r^{2k} \, dx, \qquad \eta = \frac{u_e r^k}{\sqrt{2\xi}} \int_0^y \rho \, dy \tag{4}$$

and seeking the tangential velocity in the form $u = u_e f_\eta$, we obtain

$$(l f_{\eta\eta})_\eta + f f_{\eta\eta} + \Lambda(\rho_e/\rho - f_\eta^2) = 2\xi \left(f_\eta f_{\eta\xi} - f_\xi f_{\eta\eta}\right), \qquad \Lambda = 2 \, d \ln u_e / d \ln \xi,$$

$$(l S_i^{-1} c_{i\eta})_\eta + f c_{i\eta} = 2\xi \left(f_\eta c_{i\xi} - f_\xi c_{i\eta}\right), \qquad S_i \equiv \mu/\rho D_i \quad (i = 1, \dots, N),$$

$$(l_1\sigma^{-1}g_\eta)_\eta + fg_\eta + \left\{\frac{l}{\sigma}\sum_{k=1}^{N}\frac{h_k}{H_e}(L_k-1)c_{k\eta}\right\}_\eta = 2\xi\left(f_\eta g_\xi - f_\xi g_\eta\right), \qquad (5)$$

$$l_1 = l\left[1+(\sigma-1)\frac{u_e^2}{H_e}\frac{f_\eta f_{\eta\eta}}{g_\eta}\right], \qquad l=\frac{\mu\rho}{\mu_0\rho_0}, \qquad \sigma=\frac{\mu c_p}{\lambda}, \qquad L_i=\frac{\sigma}{S_i}, \qquad g=\frac{h+u^2/2}{H_e},$$

$$\rho v = -\frac{r^{-k}}{\sqrt{2\xi}}\left[(f+2\xi f_\xi)\xi_x + 2\xi f_\eta \eta_x\right],$$

where the subscript 0 refers to conditions at the wall; $k=0$ is the plane case; $k=1$ is the axisymmetric case; u_e is the velocity of the inviscid flow outside the boundary layer; r is the radius of the transverse section of the body; H_e is the total enthalpy in the external flow. Establishing an analogy between the mass-transfer coefficients requires determination of the quantities $c'_{i\eta}(\xi,0)$. We write the boundary conditions for the functions c_i in the form

$$c_i(\xi,0)=c_{i0}(\xi), \qquad c_i(\xi,\infty)=c_{ie}=\text{const}. \qquad (6)$$

As $\xi\to 0$, the functions c_i must pass into the known self-similar solution in the neighborhood of the critical point (line). It can then be shown that the diffusion equation has only one solution for $c_{i\eta}(\xi,\eta)$ which tends exponentially to zero as $\eta\to\infty$, since the function f must satisfy the boundary conditions

$$f(\xi,0)=f_0(\xi), \qquad f_\eta(\xi,0)=0, \qquad f_\eta(\xi,\infty)=1. \qquad (7)$$

Passing to the new independent variable

$$\eta_i = \int_0^\eta \frac{S_i}{l}\,d\eta$$

(assuming that $S_i>0$), we obtain the diffusion equations in the form

$$\frac{\partial^2 c_i}{\partial\eta_i^2} + f_i\frac{\partial c_i}{\partial\eta_i} = P_i(\xi,\eta_i), \qquad (8)$$

$$P_i(\xi,\eta_i) = 2\xi\left(\frac{\partial f_i}{\partial\eta_i}\frac{\partial c_i}{\partial\xi} - \frac{\partial f_i}{\partial\xi}\frac{\partial c_i}{\partial\eta_i}\right), \qquad f_i(\xi,\eta_i)=f[\xi,\eta(\eta_i)].$$

Integrating equations (8) as inhomogeneous linear ordinary differential equations with respect to $c_{i\eta_i}$, where the functions $f_i(\xi,\eta_i)$, $P_i(\xi,\eta_i)$ are regarded as known, we obtain

$$\partial c_i/\partial\eta_i = (\partial c_i/\partial\eta_i)_{\eta_i=0}\,\Phi_i(\xi,\eta_i)\,e^{-F_i}. \tag{9}$$

Here the functions F_i and Φ_i are defined by the expressions

$$F_i(\xi,\eta_i) = \int_0^{\eta_i} f_i(\xi,\eta_i)\,d\eta_i, \qquad \Phi_i(\xi,\eta_i) = 1 + \left(\frac{\partial c_i}{\partial\eta_i}\right)^{-1}_{\eta_i=0}\int_0^{\eta_i} e^{F_i}P_i\,d\eta_i.$$

In (9), $\Phi_i(\xi,\eta_i)$ is a slowly varying function, which follows from the asymptotic solution of equations (8) for large η_i. Repeated integration of (9), taking into account the boundary conditions at infinity, gives

$$\left(\frac{\partial c_i}{\partial\eta}\right)_{\eta=0} = \frac{c_{ie}-c_{i0}(\xi)}{\omega(\infty,S_i)}S_{i0}, \qquad \omega(\infty,S_i) = \int_0^{\infty}\Phi_i(\xi,\lambda_i)e^{-F_i(\xi,\lambda_i)}\,d\lambda_i. \tag{10}$$

We shall evaluate the Laplace-type integral $\omega(S_i)$ by an asymptotic method. Let the function $f_i(\xi,\eta_i)$ be expandable in a series in η_i in a neighborhood of the line $\eta_i=0$, with coefficients depending on ξ:

$$f_i(\xi,\eta_i) = (f_i)_{\eta_i=0} + (\partial f_i/\partial\eta_i)_{\eta_i=0}\eta_i + (\partial^2 f_i/\partial\eta_i^2)_{\eta_i=0}\eta_i^2/2! +$$

$$+(\partial^3 f_i/\partial\eta_i^3)_{\eta_i=0}\eta_i^3/3! + \cdots.$$

Using the relation between the coordinates η and η_i and the boundary conditions for the function $f(\xi,\eta)$, we find

$$f_i(\xi,0) = f_0(\xi), \quad (\partial f_i/\partial\eta_i)_{\eta_i=0} = 0, \quad (\partial^2 f_i/\partial\eta_i^2)_{\eta_i=0} = \tau S_{i0}^{-2}, \quad \tau(\xi) = (\partial^2 f/\partial\eta^2)_{\eta=0},$$

$$(\partial^3 f_i/\partial\eta_i^3)_{\eta_i=0} = (\partial^3 f/\partial\eta^3)_{\eta=0}S_{i0}^{-3} + 3\tau S_{i0}^{-2}(l/S_i)_{\eta=0} = b_i.$$

Consequently,

$$F_i = \int_0^{\eta_i} f_i\,d\eta_i = f_0\eta_i + Z(\eta_i), \tag{11}$$

where

$$Z(\eta_i) = \eta_i^3 \sum_{k=0}^{\infty} a_k \eta_i^k, \qquad a_0 = \frac{\tau}{S_{i0}^2\, 3!}, \qquad a_1 = \frac{b_i}{4!}, \dots, \qquad a_k = \frac{f_{\eta_i}^{(k+2)}(\xi, 0)}{(k+3)!}.$$

Let us pass under the integral sign in formula (10) to the new integration variable $Z = F_i - f_0 \eta_i$. We have

$$\omega(\infty, S_i) = \int_0^{\infty} e^{-f_0 \lambda_i} \Phi_i[\xi, \lambda_i(Z)] \frac{d\lambda_i}{dZ} e^{-Z}\, dZ. \tag{12}$$

Here we have separated out the exponential $\exp(-f_0 \lambda_i)$, since $f_0(\xi)$ may tend to zero at some points of the surface and, in addition, since $f_0(\xi) < 0$ and $a_0(\xi) > 0$, the function $F_i(\xi, \lambda)$ in integral (10) has an additional stationary point in the coordinate λ, $\lambda \sim (-f_0/a_0)^{1/2}$. The function $Z(\lambda)$, however, has an absolute minimum at $\lambda = 0$. Inverting series (11), we obtain

$$\eta_i = Z^{1/3} \sum_{k=0}^{\infty} b_k Z^{k/3}, \qquad b_0 = a_0^{-1/3}, \qquad b_1 = -\frac{1}{3} a_1 a_0^{-5/3}, \dots,$$

whence

$$e^{-f_0 \lambda_i} \Phi_i(\xi, \lambda_i) \frac{d\lambda_i}{dZ} = \sum_{k=0}^{\infty} d_k Z^{(k-2)/3}, \tag{13}$$

where

$$d_0 = \frac{1}{3} a_0^{-1/3}, \qquad d_1 = -\frac{1}{3} a_0^{-2/3} \left[\frac{2}{3} \frac{a_1}{a_0} + f_0 + 2\xi \left(\frac{df}{d\xi} \right)_{\eta=0} \right], \dots$$

Substituting expression (13) into (12) and integrating term by term, we obtain

$$\omega(\infty, S_i) \sim \sum_{k=0}^{\infty} d_k \Gamma\left(\frac{k+1}{3} \right), \tag{14}$$

where Γ is Euler' s gamma function. It follows that the mass-transfer coefficient of the i-th component, under the condition that $S_{i0} \to \infty$ and

$$f_0 + 2\xi (\partial f / \partial \xi)_{\eta=0} \sim S_{i0}^{-1},$$

has the form

$$\frac{1}{c_{ie}-c_{i0}(\xi)}\left(\frac{\partial c_i}{\partial \eta}\right)_{\eta=0} = \frac{3S_{i0}^{1/3}}{\Gamma(1/3)}\left(\frac{\tau}{6}\right)^{1/3}\left\{1+\frac{\Gamma(2/3)}{\Gamma(1/3)}\left(\frac{6}{\tau}\right)^{1/3}\left[\alpha S_{i0}^{2/3}+\frac{1}{2}\left(\frac{l}{S_i}\right)'_{\eta=0} S_{i0}^{2/3} - \frac{1}{6}\left(\alpha+l'_{\eta=0}+\Lambda\frac{\rho_e}{\rho_0\tau}\right)S_{i0}^{-1/3}+\cdots\right]\right\}. \tag{15}$$

where the quantity $\alpha = f_0(\xi) + 2\xi(\partial l/\partial\xi)_{\eta=0}$, as follows from the last equation of system (5), is proportional to the mass blowing velocity.

The sought analogy between the mass-transfer coefficients is found in the form

$$\frac{c'_{i\eta}(\xi,0)}{c'_{j\eta}(\xi,0)} = \frac{c_{ie}-c_{i0}(\xi)}{c_{je}-c_{j0}(\xi)}\left(\frac{S_{i0}}{S_{j0}}\right)^{1/3} I(S_{i0},S_{j0}), \tag{16}$$

where

$$I(S_{i0},S_{j0}) = 1 + 0.506(6/\tau)^{1/3}\Big\{\alpha(S_{i0}^{2/3}-S_{j0}^{2/3}) + \frac{1}{2}[(l/S_i)'_{\eta=0}S_{i0}^{2/3} - (l/S_j)'_{\eta=0}S_{j0}^{2/3}] - \frac{1}{6}(\alpha+l'_{\eta=0}+\Lambda\rho_e/\rho_0\tau)(S_{i0}^{-1/3}-S_{j0}^{-1/3})+\cdots\Big\}. \tag{17}$$

Taking account of the subsequent terms in the asymptotic expansion (15) shows that the dependence of the mass-transfer coefficient on the derivative $\Lambda'(\xi)$ appears in the fifth term, and the dependence on τ'_ξ in the fourth term. This shows that relations (16) depend only weakly on the pressure gradient along the surface, if the latter varies only weakly as a function of ξ. This fact confirms the well-known "local-similarity" hypothesis in boundary-layer theory, when, as solutions of the complete equations (5), one takes the solutions of these equations without the right-hand sides, with the corresponding values of the parameter $\Lambda(\xi)$ and of the boundary conditions in the section $\xi = \text{const}$. Moreover, the structure of relations (16) coincides exactly with the corresponding expressions established for self-similar solutions [1], where instead of α one should put $\alpha(\xi) = f_0 + 2\xi(\partial f/\partial\xi)_{\eta=0}$. Therefore we may use, for further practical application, the convenient approximation of the right-hand sides of relations (16), established in [1] on the basis of a comparison of the mass-transfer coefficients with numerical solutions. Namely,

$$\frac{c'_{i\eta}(\xi,0)}{c'_{j\eta}(\xi,0)} = \frac{c_{ie}-c_{i0}}{c_{je}-c_{j0}}\left(\frac{S_{i0}}{S_{j0}}\right)^{k}, \tag{18}$$

where $k = 0.4$, if $\alpha(\xi) = 0$ and $0.25 < S_{i0}$, $S_{j0} < 5$; $k = 0$, if $0.2 < -\alpha(\xi) < 0.6$ and $0.3 < S_{i0}$, $S_{j0} < 3$; $k = -1$, if $-\alpha(\xi) \sim 1$ and $0.6 < S_{i0}$, $S_{j0} < 1.4$. Use of the generalized analogy together with definitions (1) gives a system of algebraic equations for determining the effective diffusion coefficients at the wall [2].

2. For the case of a gas mixture with close heat capacities, or when a component with a sharply different heat capacity appears in the mixture but in a small amount (for example, hydrogen in the combustion of plastics [2]), the sum on the left-hand side of the energy equation (5) may be neglected. Then, for

$$(\sigma-1)\frac{u_e^2}{H_e} \ll 1,$$

by an analogous method we obtain a generalized similarity between the mass-transfer coefficients and the heat-transfer coefficient in the form

$$\frac{c'_{i\eta}(\xi,0)}{H'(\xi,0)} = \frac{c_{ie}-c_{i0}(\xi)}{H_e-H_0(\xi)} L_{i0}^{-k}, \qquad L_i = \frac{\sigma}{S_i},$$

where the exponent k depends on the magnitude of the blowing parameter $\alpha(\xi)$.

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.