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Abstract

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MATHEMATICS

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APPROXIMATION OF CONTINUOUS FUNCTIONS BY S. N. BERNSTEIN POLYNOMIALS

(Presented by Academician S. N. Bernstein on 27 I 1964)

For estimating the approximation of an arbitrary function $f(x)$, continuous on the segment $0 \leqslant x \leqslant 1$, by means of the S. N. Bernstein polynomials

$$B_n(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) b_k(x) \quad (b_k(x) = C_n^k x^k (1-x)^{n-k}) \tag{1}$$

the estimate, valid on this segment for any natural n, is

$$|f(x) - B_n(x)| \leqslant \frac{5}{4}\,\omega\left(\frac{1}{\sqrt{n}}\right), \tag{2}$$

where here and below $\omega(\delta)$ denotes the modulus of continuity of $f(x)$. Inequality (2) was established by Popoviciu ([1]). Under the same assumptions, Li Wen-ching obtained the more precise estimate ([2])

$$|f(x) - B_n(x)| \leqslant \frac{19}{16}\,\omega\left(\frac{1}{\sqrt{n}}\right). \tag{3}$$

Below an upper estimate is obtained for the best asymptotic (as $n \to \infty$) value S_n for

$$\max_{[0,1]} |f(x) - B_n(x)| \sim S_n.$$

Keeping in mind the identity

$$\sum_{k=0}^{n} b_k(x) \equiv 1, \tag{4}$$

we successively find

$$|f(x) - B_n(x)| \leqslant \sum_{k=0}^{n} \left| f(x) - f\left(\frac{k}{n}\right) \right| b_k(k) \leqslant \sum_{k=0}^{n} \omega\left(\left| x - \frac{k}{n} \right| \right) b_k(x). \tag{5}$$

Denote by v the integer part of the number $\frac{|x-k/n|}{\delta}$, where δ denotes an arbitrary positive number,

$$v = \mathrm{E}\left(\frac{|x-k/n|}{\delta}\right). \tag{6}$$

It follows that

$$v \leqslant \frac{|x-k/n|}{\delta} < v+1. \tag{7}$$

From (7) we find

$$|x-k/n| < (v+1)\delta,$$

and, by the property of $\omega(\delta)$, we have

$$\omega(|x-k/n|) \leqslant \omega[(v+1)\delta] \leqslant (v+1)\omega(\delta).$$

With the aid of (5) and the last inequality we find

$$|f(x) - B_n(x)| \leqslant \omega(\delta) \sum_{k=0}^{n} \left[1 - v\left(x, \frac{k}{n}\right)\right] b_k(x).$$

The last inequality, using (4), can be put in the form

$$|f(x) - B_n(x)| \leqslant \omega(\delta) \left[1 + \sum_{k=0}^{n} v\left(x, \frac{k}{y}\right) b_k(x)\right]. \tag{8}$$

Next we estimate

$$\sum_{k=0}^{n} v\left(x, \frac{k}{n}\right) b_k(x). \tag{9}$$

From the left-hand side of the double inequality (7) we obtain

$$v\left(x, \frac{k}{n}\right) \leqslant \frac{|x-k/n|^{\alpha}}{\delta^{\alpha}}, \tag{10}$$

where $\alpha \geqslant 1$. From (9) and (10) we have

$$\sum_{k=0}^{n} v\left(x, \frac{k}{n}\right) b_k(x) < \frac{1}{\delta^{\alpha}} \sum_{k=0}^{n} \left| x - \frac{k}{n} \right|^{\alpha} b_k(x). \tag{11}$$

Let us find the asymptotic value of the right-hand side of (11) as $n \to \infty$. For this purpose it is convenient to represent the sum on the right-hand side of (11) in the form of an integral over a simple piecewise-smooth closed contour C, which in the complex z-plane encloses the segment $[0, 1]$ of the real axis Ox:

$$\sum_{k=0}^{n} \left|x - \frac{k}{n}\right|^{\alpha} b_k(x) = \frac{n!}{2\pi i} \int_C |z - x|^{\alpha} \frac{(x-1)^{n(1-z)} x^{nz}}{z(nz-1)(nz-2)\dots(nz-n)}\, dz.$$

As a result of applying Stirling’ s formula we obtain

$$\sum_{k=0}^{n} \left|x - \frac{k}{n}\right|^{\alpha} b_k(x) = \sqrt{\frac{n}{2\pi}} \int_C (1+\rho_n')|z-x|^{\alpha} \frac{1}{\sqrt{z(z-1)}} \left[\left(\frac{x-1}{z-1}\right)^{1-z} \left(\frac{x}{z}\right)^{z}\right]^{n} dz, \tag{12}$$

where here $n\rho_n'$ remains bounded as $n \to \infty$, and, moreover, we assume $0 < x < 1$ ([3]).

Applying to the right-hand side of (12) Perron’ s method ([4]) for finding the asymptotic value of integrals, we obtain, as $n \to \infty$ and $0 < x < 1$,

$$\sum_{k=0}^{n} \left|x - \frac{k}{n}\right|^{\alpha} b_k(x) \sim \frac{1}{\sqrt{\pi}} [2x(1-x)]^{\alpha/2} \Gamma\left(\frac{\alpha+1}{2}\right) \frac{1}{n^{\alpha/2}}.$$

Hence we conclude that, in the interval $0 < x < 1$,

$$\max_{(0,1)} \sum_{k=0}^{n} \left|x - \frac{k}{n}\right|^{\alpha} b_k(x) \sim \frac{1}{\sqrt{\pi}} \frac{\Gamma((\alpha+1)/2)}{2^{\alpha/2}} \frac{1}{n^{\alpha/2}}. \tag{13}$$

Thus, from (8), (11), and (13) we obtain on the segment $0 \leqslant x \leqslant 1$

$$\max_{[0,1]} |f(x) - B_n(x)| \sim S_n \leqslant \left[1 + \frac{1}{\sqrt{\pi}} \frac{\Gamma((\alpha+1)/2)}{2^{\alpha/2}} \frac{1}{(\delta\sqrt{n})^{\alpha}}\right] \omega(\delta),$$

since $f(0) - B_n(0) = 0$ and $f(1) - B_n(1) = 0$.

Putting here $\delta = 1/\sqrt{n}$, we obtain

$$\max_{[0,1]} |f(x) - B_n(x)| \sim S_n \leqslant \left[1 + \frac{1}{\sqrt{\pi}} \frac{\Gamma((\alpha+1)/2)}{2^{\alpha/2}}\right] \omega\left(\frac{1}{\sqrt{n}}\right). \tag{14}$$

Put in (14) $\alpha = 2$; then

$$\max_{[0,1]} |f(x) - B_n(x)| \sim \frac{5}{4}\, \omega\left(\frac{1}{\sqrt{n}}\right) = 1.25\, \omega\left(\frac{1}{\sqrt{n}}\right);$$

this is a result due to Popoviciu. If, however, we put $\alpha = 4$, then we obtain

$$\max_{[0,1]} |f(x) - B_n(x)| \sim \frac{19}{16}\,\omega\left(\frac{1}{\sqrt{n}}\right) = 1.1875\,\omega\left(\frac{1}{\sqrt{n}}\right);$$

this is the estimate obtained by Li Wen-ching.

Having in the right-hand side of (14) a general expression for arbitrary $\alpha \geqslant 1$, it is not difficult to obtain for the constant

$$1 + \frac{1}{\sqrt{\pi}}\,\frac{\Gamma((\alpha+1)/2)}{2^{\alpha/2}} \tag{15}$$

the best value. For this it is necessary to find such a number $\alpha = \xi$ for which expression (15) is the least. This number ξ is the unique positive root of the equation

$$\frac{\Gamma'((\xi+1)/2)}{\Gamma((\xi+1)/2)} - \ln 2 = 0. \tag{16}$$

Solving this equation, we find

$$\xi = 2.958\ldots.$$

Consequently, we have the upper estimate of the best asymptotic (as $n \to \infty$) value S_n

$$\max_{[0,1]} |f(x) - B_n(x)| \sim S_n \leqslant \left[1 + \frac{1}{\sqrt{\pi}}\,\frac{\Gamma((\xi+1)/2)}{2^{\xi/2}} + o(1)\right]\omega\left(\frac{1}{\sqrt{n}}\right) =$$

$$= [1.1761\ldots + o(1)]\,\omega\left(\frac{1}{\sqrt{n}}\right). \tag{17}$$

One may suppose that

$$S_n \approx \left[1 + \frac{1}{\sqrt{\pi}}\,\frac{\Gamma((\xi+1)/2)}{2^{\xi/2}}\right]\omega\left(\frac{1}{\sqrt{n}}\right)$$

and, moreover,

$$|f(x) - B_n(x)| \leqslant \left[1 + \frac{1}{\sqrt{\pi}}\,\frac{\Gamma((\xi+1)/2)}{2^{\xi/2}}\right]\omega\left(\frac{1}{\sqrt{n}}\right)$$

is fulfilled for $0 \leqslant x \leqslant 1$ for all natural n.

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.