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Abstract

Full Text

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ON A CLASS OF FINITE HOMOMORPHISMS

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Let $h(n)$ be a finite homomorphism of the natural sequence, i.e. a completely multiplicative function of a natural argument whose values form a finite set of complex numbers, at least one of which is different from zero; let m be the number of nonzero values of the function $h(n)$;

$$S(x) = \sum_{n \le x} h(n)$$

is the summatory function of the finite homomorphism $h(n)$; p, q, with or without subscripts, are prime numbers; $\chi_0(n,k)$ is the principal Dirichlet character of modulus k; $\varphi(n)$ is Euler' s function; $\zeta(s)$ is the Riemann zeta-function; $s = \sigma + it$.

A finite homomorphism will be called a **generalized character** if its summatory function satisfies the condition $S(x) = \alpha x + O(1)$ with some constant α, generally speaking complex; a generalized character is called **principal** if $\alpha \neq 0$, and **nonprincipal** if $\alpha = 0$. The so-called problem of characters, not solved up to now, is as follows: do there exist (apart from one trivial example) generalized characters distinct from Dirichlet characters? (see ([1,2])). In the case of principal characters the answer to this question is given by the following theorem.

Theorem. *Every principal generalized character is a principal Dirichlet character.*

Basic lemma. *If $h(n)$ is not a Dirichlet character and the series*

$$\sum_{h(p) \neq 1} \frac{1}{p}$$

converges, then $h(n)$ cannot be a principal generalized character.

Lemma 1. *The product of a principal generalized character by a principal Dirichlet character $\chi_0(n,k)$ is again a principal generalized character.*

The proof is obtained without difficulty by induction on the number of primes entering into k (cf. ([1])).

Lemma 2. *Let $h(n)$ not be a Dirichlet character and let $h(p) \neq 1$ for only a finite set of primes p. Then $h(n)$ cannot be a generalized character.*

Proof. Let $h(p) \neq 1$ for the primes $p_1, p_2, \dots, p_r$. Since $h(n)$ is not a Dirichlet character, for some $i \leq r$ we have $h(p_i) \neq 1, 0$. We shall assume that $h(p_1) = \lambda \neq 1, 0$. Suppose now that $S(x) = \alpha x + O(1)$. Putting $h_1(n) = h(n)\chi_0(n, p_2, p_3 \dots p_r)$, we shall have, by Lemma 1,

$$S_1(x) = \sum_{n \leq x} h_1(n) = \alpha_1 x + O(1) \tag{1}$$

with some constant α_1. Choose a prime $p \neq p_i$ $(i = 1, 2, \dots, r)$ and consider the functions

$$f(n) = \begin{cases} \chi_0(n, p_2 \dots p_r), & \text{if } p \nmid n, \\ (1-p)\chi_0\left(\dfrac{n}{p}, p_2 \dots p_r\right), & \text{if } p \mid n; \end{cases}$$

$$\tilde{f}(n) = \begin{cases} f(n), & \text{if } p_1 \nmid n, \\ \lambda^\beta f\left(\dfrac{n}{p_1^\beta}\right), & \text{if } p_1^\beta \parallel n. \end{cases}$$

Applying to the pair of functions $f(n)$, $\tilde{f}(n)$ literally the same arguments as in the proof of Theorem 1 of paper $(^3)$, we obtain the unboundedness of the summatory function $\sum_{n \leq x} \tilde{f}(n)$. On the other hand, evidently,

$$\tilde{f}(n) = h_1(n), \quad \text{if } p \nmid n, \qquad \tilde{f}(np) = (1-p)h_1(n).$$

Hence

$$\sum_{n \leq x} \tilde{f}(n) = \sum_{\substack{n \leq x \\ p \nmid n}} \tilde{f}(n) + \sum_{n \leq x/p} \tilde{f}(np) = \sum_{n \leq x} h_1(n) - \sum_{n \leq x/p} h_1(np) + \sum_{n \leq x/p} (1-p)h_1(n) = S_1(x) - S_1\left(\frac{x}{p}\right) + (1-p)S_1\left(\frac{x}{p}\right)$$

by virtue of relation (1). The contradiction obtained proves the lemma.

Lemma 3. *The principal generalized character is different from zero at all primes, with the possible exception of only finitely many of them.*

Proof. Suppose $h(p_i) = 0$ for an infinite sequence of primes $p_1, p_2, \dots$, and at the same time $S(x) = ax + O(1)$, where $\alpha \neq 0$. Then find a natural y from the system of congruences $y + i \equiv 0 \pmod{p_i}$, $i = 1, 2, \dots, t$, where t is any natural number. Evidently, $h(y+i) = 0$, and therefore $S(y+t) = S(y)$, whence $\alpha t = O(1)$, which is impossible for $\alpha \neq 0$ and arbitrary t.

Proof of the main lemma will be carried out by contradiction. Suppose $h(p_i) \neq 1$ on a set of primes $p_1, p_2, \dots$ such that the series $\sum_i 1/p_i$ converges, and $S(x) = \alpha x + O(1)$, where $\alpha \neq 0$. Obviously, using Lemmas 2, 3, and 1, we see that one may assume the following additional conditions to be satisfied: 1) the sequence $p_1, p_2, \dots$ is infinite; 2) $h(p_i) = 0$ only for a finite number of primes p_i; denote their product by k; 3) for each of the m nonzero values of $h(n)$, the set of primes for which $h(p)$ is equal to this value is either empty or infinite. Assuming all the listed conditions to be satisfied, put

$$m = q_1^{\delta_1} q_2^{\delta_2} \cdots q_u^{\delta_u}$$

—the canonical factorization of the number m; let $\xi_1, \xi_2, \dots, \xi_u$ be fixed primitive roots of unity of orders $q_1^{\delta_1}, q_2^{\delta_2}, \dots, q_u^{\delta_u}$, respectively. It is clear that any nonzero value $h(n)$ is represented in a unique way in the form

$$h(n) = \prod_{i=1}^{u} \xi_i^{\varepsilon_i(n)},$$

where $0 \leq \varepsilon_i(n) < q_i^{\delta_i}$. We shall suppose the primitive roots $\xi_1, \dots, \xi_u$ chosen so that from the sequence $p_1, p_2, \dots$ one can extract u disjoint infinite subsequences $p_{i1}, p_{i2}, \dots$ $(i = 1, 2, \dots, u)$ with the condition: $\varepsilon_i(p_{ij}) = 1$ for $i = 1, 2, \dots, u$ and all j. Such a choice is always possible.

Next, let $\eta = \xi_i^{\varepsilon}$, where ε is equal to 0 or 1 and is chosen so that $\alpha \neq \eta\varphi(k)/k$; a is any natural number exceeding $2m/k$. Consider all natural

$$n \leq ak.$$

If $(n, k) = 1$, then

$$h(n) = \prod_{i=1}^{u} \xi_i^{\varepsilon_i(n)},$$

whereas if $(n, k) > 1$, then $h(n) = 0$. Choose a natural ν so large that between ak and p_ν, in each of the u subsequences $p_{i1}, p_{i2}, \dots$, there are no fewer than $a\varphi(k)$ primes and

$$\sum_{i=\nu+1}^{\infty} \frac{1}{p_i} < \frac{1}{2ak}. \tag{2}$$

Take from each subsequence $a\varphi(k)$ primes lying between ak and p_ν, and put them in one-to-one correspondence with the natural numbers $n \leq ak$,

coprime to k; the number from the i-th subsequence $p_{i1}, p_{i2}, \dots$ corresponding to the given n will be denoted by $\bar{p}_{ni}$. As already stated, $ak < \bar{p}_{ni} < p$, $\varepsilon_i(\bar{p}_{ni}) = 1$.

Consider the obviously solvable system of $a\varphi(k)+1$ congruences

$$x \equiv 0 \left(\bmod \prod_{j=1}^{\nu}{}' p_j^{ak}\right), \tag{A}$$

$$x + n \equiv \prod_{i=1}^{u} \bar{p}_{ni}^{\,m q_i^{-\delta_i} \beta_i(n)} \left(\bmod \prod_{i=1}^{u} \bar{p}_{ni}^{ak}\right), \quad n \le ak, \quad (n,k)=1,$$

where $\prod'$ denotes the product over primes distinct from all $\bar{p}_{ni}$, and $q_i^{\beta_i(n)}$ is the least natural number satisfying the congruence

$$\frac{m}{q_i^{\delta_i}} x + \varepsilon_i(n) \equiv \omega_i \pmod{q_i^{\delta_i}}; \quad \omega_1 = \varepsilon, \quad \omega_2 = \omega_3 = \dots = \omega_u = 0.$$

If b is the least natural number satisfying the system (A), then it is satisfied by all numbers of the form $b + Qy$, where $Q = \prod_{j=1}^{\nu} p_j^{ak}$.

It is now not hard to observe that the numbers

$$b+1+Qy,\ b+2+Qy, \dots,\ b+ak+Qy \tag{3}$$

for any natural y have the form

$$b+n+Qy = p_1^{\gamma_{1n}} p_2^{\gamma_{2n}} \cdots p_\nu^{\gamma_{\nu n}} \cdot b_n(y) = a_n b_n(y), \tag{4}$$

where $(b_n(y), Q) = 1$, a_n does not depend on y, and $h(a_n) = \eta\chi_0(n,k)$.

We shall show that there exists a natural y for which all the numbers (3) are coprime to $p_{\nu+1}, p_{\nu+2}, \dots$. Let $N(z)$ denote the number of natural y, not exceeding z, for which at least one of the numbers (3) is divisible by one of $p_{\nu+1}, p_{\nu+2}, \dots$. Since $p_{\nu+i} > ak$, among the numbers (3) no more than one can be divisible by a given $p_{\nu+i}$. Hence, when y runs through a complete system of residues modulo $p_{\nu+i}$, among the sets of the form (3) there will be exactly ak such sets in which one of the numbers is divisible by $p_{\nu+i}$. It follows that, for $z \ge 2$, we obtain:

$$N(z) \le ak \sum_{p_\nu < p_i \le 2Qz} \left(\left[\frac{z}{p_i} \right] + 1 \right) < akz \sum_{i=\nu+1}^{\infty} \frac{1}{p_i} + ak\pi(2Qz).$$

The first term on the right-hand side is less than $z/2$ by virtue of inequality (2), and the second will be less than $z/3$ if we choose $z = e^{12akQ}$. Hence we obtain $N(z) < \frac{5}{6} z \le z - 2$. Consequently, for some $y \le e^{12akQ}$ all the numbers (3), and therefore all the numbers $b_n(y)$ in the representation (4), are coprime to $p_{\nu+1}, p_{\nu+2}, \dots$. But $(b_n(y), Q) = 1$, therefore $(b_n(y), p_i) = 1$ also for $i = 1, 2, \dots,$

whence $h(b_n(y)) = 1$, i.e. $h(b+n+Qy) = \eta\chi_0(n,k)$ for $n = 1, 2, \dots, ak$. It follows that for the summatory function we obtain

$$S(b+ak+Qy) - S(b+Qy) = \sum_{n=1}^{ak} h(b+n+Qy) = \eta a\varphi(k).$$

On the other hand, this difference is $aak + O(1)$, since we assumed that $S(x) = ax + O(1)$. Hence $\eta a\varphi(k) = aak + O(1)$. But this is impossible by the choice of η and the arbitrariness of a. The lemma is proved.

Proof of the theorem. Consider the function defined, for $\sigma > 1$, by the Dirichlet L-series for $h(n)$, and apply Abel' s formula to this series. We obtain

$$L(s) = \sum_{n=1}^{\infty} \frac{h(n)}{n^s} = -s\int_1^{\infty} \frac{S(x)\,dx}{x^{s+1}} = \alpha\,\frac{s}{s-1} + O\left(\frac{|s|}{\sigma}\right), \tag{5}$$

since $S(x) = \alpha x + O(1)$. From equality (5) we conclude that $L(s)$ is regular in the entire half-plane $\sigma > 0$, except for a pole of the first order at the point $s = 1$. Hence the function $g(s) = \zeta(s)/L(s)$ is regular in some neighborhood of the point $s = 1$. For $\sigma > 1$, expansion into an Euler product gives

$$g(s) = \prod_p \left(1 - \frac{h(p)}{p^s}\right)\left(1 - \frac{1}{p^s}\right)^{-1} = \prod_{h(p)\neq 1} \left(1 - \frac{h(p)}{p^s}\right)\left(1 - \frac{1}{p^s}\right) =$$

$$= \prod_{h(p)\neq 1} \left(1 + \frac{1-h(p)}{p^s - 1}\right).$$

Let us note that, for $\sigma > 1$ and $h(p) \neq 1$,

$$\left|1 + \frac{1-h(p)}{p^\sigma - 1}\right|^2 \geq 1 + \frac{1 - \operatorname{Re} h(p)}{p^\sigma} \geq 1 + \frac{c}{p^\sigma},$$

where the constant $c > 0$ depends only on m. Therefore, for $\sigma > 1$,

$$|g(\sigma)|^2 = \prod_{h(p)\neq 1} \left|1 + \frac{1-h(p)}{p^\sigma - 1}\right|^2 \geq \prod_{h(p)\neq 1} \left(1 + \frac{c}{p^\sigma}\right) > c \sum_{h(p)\neq 1} \frac{1}{p^\sigma}.$$

If the last series diverged at $\sigma = 1$, then its sum would tend to infinity as $\sigma \to 1+0$. But then, as $\sigma \to 1+0$, $|g(\sigma)| \to \infty$, which contradicts the continuity of $g(s)$ at the point $s = 1$. Hence the series

$$\sum_{h(p)\neq 1}\frac{1}{p}$$

converges, and our principal generalized character $h(n)$ is a Dirichlet character by virtue of the main lemma. The theorem is proved.

Remark 1. The condition of generality of the character, i.e. $S(x)=\alpha x+O(1)$, cannot be weakened. It can be shown that, for any function $F(x)$ increasing arbitrarily slowly to infinity, there exists a finite homomorphism, distinct from a Dirichlet character, whose summatory function is $\alpha x+\Omega(F(x))$ with $\alpha\neq 0$.

Remark 2. The main lemma remains valid also in the case of nonprincipal characters in the following form: if $h(n)$ is not a Dirichlet character, then for some nonprincipal Dirichlet character $\chi(n)$ the series

$$\sum_{(hp)\neq\chi(p)}\frac{1}{p}$$

converges, whereas the summatory function of the finite homomorphism $h(n)$ is unbounded. The corresponding Ω-estimates can be obtained. This is much stronger than the previous results on "spoiled" characters (3,4).

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