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Abstract

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ON A PROBLEM IN APPROXIMATION THEORY CONCERNING SUPERPOSITIONS OF FUNCTIONS

(Presented by Academician S. N. Bernstein on 22 VII 1963)

Let G be a regular topological space with a countable base, and let $u = A(x)$ be a single-valued continuous operator mapping G onto some metric space Q_ρ with metric $\rho(u, v)$, and let $H(Q_\rho)$ be the class of bounded real functions satisfying on Q_ρ the Hölder condition

$$|f(u) - f(v)| \leqslant \rho(u, v). \tag{1}$$

Each operator $u = A(x)$ associates with all functions $f \in H(Q_\rho)$ the superpositions $f[A(x)]$ generated by it, which are real functions $F(x) = f[A(x)]$ defined on G. When considering two different operators $A(x)$ and $B(x)$ defined on G, the problem arises of the best uniform approximation of superpositions generated by one of these operators, for example $A(x)$, by superpositions generated by the second operator $B(x)$.

In the present note we shall give some concrete results related to this problem for the case when $A(x) = \lambda(x)$ and $B(x) = \mu(x)$ are continuous functionals defined on G and mapping this space onto the entire complex plane, or onto some part of it.

Theorem 1. *If $\lambda(x)$ and $\mu(x)$ are linear functionals defined on some linear normed space E, then for any real-valued, bounded and absolutely continuous function $f(u)$, defined in the complex plane or on the real number axis, whose derivative does not exceed one in modulus, one can indicate a function $g(u)$ with the same properties and such that everywhere on the unit sphere S of the space E the inequality*

$$|f[\lambda(x)] - g[\mu(x)]| \leqslant \sup_{\|y\| \leqslant 1} \{|\lambda(y)| - |\mu(y)|\} \tag{2}$$

holds.

This result cannot be improved, since the upper bound of the corresponding best approximations over all $x \in S$ and all functions $f(u)$ under consideration is exactly equal to the right-hand side of (2).

It is obvious that if $\operatorname{sign} \lambda(x) = \operatorname{sign} \mu(x)$, then the right-hand side of inequality (2) is equal to $\|\lambda - \mu\|$, and, consequently, in this case one may take the function $f(u)$ itself as $g(u)$, and no better approximation over the entire class of functions under consideration can be obtained. In other cases, as simple examples show, the stated result may turn out to be substantially more precise than the inequality

$$\sup_{\|x\| \leqslant 1} |f[\lambda(x)] - f[\mu(x)]| \leqslant \|\lambda - \mu\|$$

for the functions under consideration.

Alongside the obvious example $\mu(x) = -\lambda(x)$, one can give another simple example illustrating this remark, related to series

Fourier. If E is the space $\widetilde{C}$ of continuous functions of period 2π with norm $|x(t)| = \max_t |x(t)|$, $\lambda(x)$ is the mean value of $x(t)$ over a period, and

$$\mu(x) = \frac{1}{2\pi} \int_0^{2\pi} x(t) e^{int}\, dt$$

is the n-th Fourier coefficient of $x(t)$, then, for any integer $n \neq 0$, the upper bound of the corresponding best approximations of superpositions $f[\lambda(x)]$ in accuracy is equal to one.

For arbitrary (nonlinear) continuous functionals the following result holds.

Theorem 2. *If the elements of the space G with a countable base can be ordered so that both functionals $\lambda(x)$ and $\mu(x)$ are then simultaneously monotone, then, for $\theta^+(x) = \frac{1}{2}[\lambda(x) + \mu(x)]$ and $\theta^-(x) = \frac{1}{2}[\lambda(x) - \mu(x)]$, the upper bound of the best uniform approximations on G of the superpositions $f[\lambda(x)]$ by superpositions $g[\mu(x)]$, in the class of all real, bounded and absolutely continuous functions $f(u)$ and $g(u)$ whose derivative almost everywhere does not exceed one in modulus, is equal to*

$$\sup_{x \leq y} \{\theta^-(y) - \theta^-(x)\}, \quad \text{when } \lambda(x) \text{ and } \mu(x) \text{ are nondecreasing,}$$

$$\sup_{x \leq y} \{\theta^-(x) - \theta^-(y)\}, \quad \text{when } \lambda(x) \text{ and } \mu(x) \text{ are nonincreasing,}$$

$$\sup_{x \leq y} \{\theta^+(y) - \theta^+(x)\}, \quad \text{when } \lambda(x) \text{ is nondecreasing and } \mu(x) \text{ is nonincreasing,}$$

$$\sup_{x \le y}\{\theta^+(x) - \theta^+(y)\} : \quad \text{when } \lambda(x) \text{ is nonincreasing and } \mu(x) \text{ is nondecreasing.}$$

We shall note in particular one concrete example, of known interest in classical approximation theory, of the application of this result.

In the study of various questions connected with the approximation of functions defined on a finite interval of the real axis by algebraic polynomials, in approximation theory one often reduces the problem, by the substitution (in the case of the interval $[-1, 1]$) $u = \cos t$, to the study of approximations of even periodic functions by trigonometric polynomials. In this connection, alongside the class of even functions $f(t)$ of period 2π satisfying on the whole real axis the condition

$$|f(t_1) - f(t_2)| \le |t_1 - t_2|, \tag{3}$$

one also considers the narrower class of even functions $g(t)$ of period 2π satisfying the condition

$$|g(t_1) - g(t_2)| \le |\cos t_1 - \cos t_2|. \tag{4}$$

The following theorem holds, establishing the magnitude of the deviation of these two classes of functions.

Theorem 3. *The upper bound of the best uniform approximation of functions $f(t)$ satisfying condition (3) by functions $g(t)$ satisfying condition (4), over all functions of the first class, is equal to $\pi/2 - 1$.*

The proof of the theorems stated is connected with the study of the change of a class of functions, defined in a separable metric space and satisfying there a Hölder condition, in passing from a given metric $\rho(x, y)$ to some other metric $r(x, y)$.

Individual problems of this type were considered earlier by the author ([1–3]).

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CITED LITERATURE

[1] A. F. Timan, DAN, **140**, No. 2, 307 (1961).
[2] A. F. Timan, DAN, **150**, No. 1, 52 (1963).
[3] A. F. Timan, DAN, **150**, No. 2, 266 (1963).

Note: Figure translations are in progress. See original paper for figures.