CYBERNETICS AND CONTROL THEORY

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Abstract

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ON LIMITING DISTRIBUTIONS IN A STOCHASTIC MODEL OF LEARNING

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For the mathematical description of certain experiments on learning, the following scheme was proposed in (1).

The learner (hereinafter called the subject), at time τ (in the τ-th experiment), produces a response $R_j \in R = (R_1, \dots, R_r)$ with probability $p_\tau(R_j)$. In reply to it, the environment (or the experimenter) carries out an action $S_k \in S = \{S_1, \dots, S_r\}$ with probability $\pi_{kj} = p(S_k/R_j)$ ($p(S_k/R_j)$ is the conditional probability of the environmental action S_k, if the subject has made the response R_j). The entire past "experience" of the subject, all the "knowledge" about the environment accumulated by him up to time τ, is contained in the mode of his behavior—in the probability distribution of his responses

$$\mathbf{p}_\tau = (p_\tau(R_1), \dots, p_\tau(R_r)).$$

Additional "knowledge" about the environment, obtained by the subject as a result of the τ-th experiment, serves to change the probabilities of his responses and is completely contained in the event (R_j, S_k)—the subject' s response in this experiment and the environment' s reply to it. Mathematically this is expressed by specifying functions φ_{kj} mapping the set of probability measures on $R = \{R_1 \dots R_r\}$ into itself. Moreover, if $\mathbf{p}_{\tau'} = \mathbf{p}_{\tau''}$ and both experiments (the τ'-th and the τ''-th) ended with one and the same event (R_j, S_k), then

$$\mathbf{p}_{\tau'+1} = \mathbf{p}_{\tau''+1} = \varphi_{kj}(\mathbf{p}_{\tau'}) = \varphi_{kj}(\mathbf{p}_{\tau''}).$$

The response of the environment in the τ-th experiment is random, and the probability that it will end with the event (R_j, S_k) depends only on $\mathbf{p}_\tau$; therefore $\mathbf{p}_\tau$ is a homogeneous Markov chain.

Under what conditions will the subject eventually learn something? What happens to the subject under an unbounded increase of learning time? To what extent does the model agree with the experimental data?

To answer these questions it is necessary to study the asymptotic properties of the chain $\mathbf{p}_\tau$.

In (1) it is assumed that the functions φ_{kj} are linear:

$$\varphi_{kj} = \alpha_{kj} E + (1 - \alpha_{kj}) Q_{kj},$$

where $0 \leq \alpha_{kj} \leq 1$, E is the identity matrix, and Q_{kj} is a stochastic matrix whose rows are the vector

$$\mathbf{q}_{kj} = (q_{kj1}, \dots, q_{kjr}).$$

Suppose that the set of the subject' s responses consists of two elements, $R = \{R_1, R_2\}$, and denote $p_\tau(R_1) = p_\tau$, $q_{kj1} = q_{kj}$, $\alpha_{kj} = (1 - a_{kj})q_{kj}$, $M_{m,\tau}$ the moment of order m of the random variable p_τ.

Consider three types of environmental responses:

a) the environmental responses do not depend on the subject' s responses

$$(\pi_{kj} = \pi_k, \quad a_{kj} = a_k, \quad q_{kj} = q_k);$$

b) the environmental responses are uniquely determined by the subject' s responses

$$(S = (S_1, S_2), \quad \pi_{kj} = \delta_{kj}, \quad a_{kj} = a_k, \quad q_{kj} = q_k);$$

c) the environmental responses depend stationarily and stochastically on the subject' s responses

$$(\pi_{kj} = p(S_k / R_j)).$$

For each of these three types of environmental responses, formulas are easily derived (see (1), § 4, 3–§ 4, 6) expressing $M_{m,\tau+1}$ in terms of $M_{n,\tau}$, $n = 1, \dots, m+1$. These formulas have, in all three cases, the form

$$\mathbf{M}_{\tau+1} = C + A\mathbf{M}_\tau = \left(\sum_{k=0}^{\tau-1} A^k \right) C + A^\tau \mathbf{M}_1, \tag{1}$$

where

$$\mathbf{M}_\tau = (M_{1,\tau}, M_{2,\tau} \dots, M_{m,\tau}, \dots); \quad C = (C_1, \dots, C_m, \dots), \quad A = \|C_{kj}\|_{k,j=1}^{\infty};$$

C_m, C_{kj} are real numbers uniquely determined through π_{kj}, a_{kj}, q_{kj}.

Let M be the set of all vectors

$$\mathbf{M} = (M_1, \dots, M_m, \dots)$$

such that,

that

$$M_m = \int_0^1 x^m \, dF(x),$$

where $F(x)$ are the distribution functions of random variables ξ equal to 0 outside $[0,1]$. M is a closed subset of the B-space of bounded sequences. It is easy to show that in case a) always, and under certain restrictions on a_{kj}, α_{kj} (and perhaps always), and in cases b) and c), A is a contraction operator on M. Passing to the limit ($\tau \to \infty$), where possible, we obtain equations for the moments of the limiting distribution in the form

$$\mathbf{M} = \mathbf{C} + A\mathbf{M}. \tag{2}$$

Hence, using elementary formulas of combinatorial analysis (see (2), Ch. 2, § 4), we obtain equations for the characteristic functions $f(t)$ of the limiting distributions:

$$f(t) = \sum_{k=1}^{s} \pi_k e^{ia_k t} f(\alpha_k t), \tag{3}$$

$$f(t) = e^{ia_2 t} f(\alpha_2 t) - \frac{ie^{ia_1 t}}{\alpha_1} f'(\alpha_1 t) + \frac{ie^{ia_2 t}}{\alpha_2} f'(\alpha_2 t), \tag{4}$$

$$f(t) = \sum_{k=1}^{s} \left\{ \pi_{k,2} e^{ia_{k,2} t} \left[f(\alpha_{k,2} t) + \frac{i}{\alpha_{k,2}} f'(\alpha_{k,2} t) \right] + \pi_{k,1} \frac{ie^{ia_{k,1} t}}{\alpha_{k,1}} f'(\alpha_{k,1} t) \right\}. \tag{5}$$

Examples.

1) Let $q_{ki} = q$; then for all three types there exists a unique limiting distribution. Equations (3)–(5) have the solution e^{iqt}, i.e. the distribution is concentrated at the point q.

2) Consider the first type of environmental responses. In (3) put $\alpha_k = \alpha$. Then

$$f(t) = f(\alpha t) \left(\sum_{k=1}^{s} \pi_k e^{ia_k t} \right).$$

Iterating this relation, we obtain

$$f(t) = f(\alpha^k t) \prod_{m=0}^{n} \left(\sum_{k=1}^{s} \pi_k e^{ia_k \alpha^m t} \right) = \prod_{m=0}^{\infty} \left(\sum_{k=1}^{s} \pi_k e^{ia_k \alpha^m t} \right).$$

This is the characteristic function of the sum $\sum_{m=0}^{\infty} \xi_m$, where ξ_m are independent random variables, and ξ_m takes the values $a_k \alpha^m$, $k = 1, \dots, s$, with probabilities π_k $(0 < \alpha < 1)$.

For particular values of the parameters this expression occurs in the problem of uniqueness of Fourier series (3). For example, for $S = \{S_1, S_2\}$, $\alpha = 1/3$, $\pi_1 = \pi_2 = 1/2$, $a_1 = 0$, $a_2 = 2/3$ (i.e. $q_1 = 0,\ q_2 = 1$),

$$f(t) = \lim_n \frac{1}{2^n} \prod_{m=0}^{n} \left(1 + e^{\frac{2it}{3^{m+1}}}\right) = e^{it} \prod_{m=0}^{\infty} \cos\left(\frac{2t}{3^{m+1}}\right) = \int_0^1 e^{itx}\, dF(x).$$

Hence ((3), p. 828) $F(x)$ is the Cantor singular curve on $[0, 1]$, and p_∞, corresponding to $F(x)$, takes values from the Cantor perfect set. It is shown in (4) that this may also occur in case b). There is reason to suppose that, in the general case, when $R = \{R_1 \dots R_r\}$, $S = \{S_1 \dots S_s\}$, $\alpha_{kj} < 1 - \alpha_{k'j'}$, p_∞ is also concentrated on a perfect, nowhere dense set without isolated points.

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Note: Figure translations are in progress. See original paper for figures.

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