CYBERNETICS AND CONTROL THEORY

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Abstract

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On a Production Model in Dynamic Programming

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In works (1, 2) a production model was considered, applied to the calculation of the best distribution of coupled investments among various branches of industry. On the basis of this model, by a limiting transition, its continuous analogue was obtained. The limiting transition, carried out there only for the first approximation, not only does not make it possible to obtain results with increased accuracy, but was carried out imprecisely. The first consequence of this, from the mathematical point of view, was that the author was forced to operate with δ-functions, which considerably complicate the apparatus employed; from the economic point of view, in the description of the model there is absent a very essential parameter of production dynamics—the duration of a complete finished production cycle.

Using the classification of operating regimes of production introduced by us, we may say that in works (1, 2) one of the two basic operating regimes of production is excluded from consideration, namely the regime in which the production capacities are sufficient and the amount of finished output produced by the production process per unit time is determined only by the amount of raw material available.

In the present work we consider a model which is likewise discrete, but differs from the one cited above (with the stated remarks taken into account). For its description, certain piecewise-smooth characteristics have been chosen and conditions imposed on these characteristics in the form of a system of differential equations and inequalities. The latter makes it possible to carry out a clear classification of the operating regimes of production and to pose optimization problems effectively (see (1, 2, 4)).

Let us consider a production model in which part of its output is exchanged for raw material. Suppose that from one unit of product allocated for exchange for raw material, over the course of τ selected units of time, the production process manufactures β units of finished product. We shall call the quantity τ the **duration of a complete finished production cycle**. For convenience of consideration, let the duration of the consumption of raw material by production (or the duration of loading) to ensure the operation of production during one production cycle be denoted by τ_1, and let the duration of delivery to the warehouse of the finished output produced during the cycle be denoted by τ_2 $(\tau_1 + \tau_2 \leqslant \tau)$.

Suppose that the rates of loading and delivery during one cycle are constant and are equal, respectively, to u_1^k and u^k units of product per unit time, where k is the cycle number $(k = 1, 2, ...)$. Let us establish that the capacity of the enterprise does not exceed the nominal value Q units of product per cycle, i.e. $\tau_2 u^k = \beta\tau_1 u_1^k \leqslant Q$ $(k = 1, 2, ...)$. In addition, suppose that from the warehouse at the end of each cycle and at the beginning of the next, from the moment $k\tau - \tau_3$ to the moment $k\tau + \tau_4$, output is shipped at a constant rate u_2^k for this interval (or is shipped at the rate $-u_2^k$) units of product per unit time $(k = 1, 2, 3, ...;\ \tau_3 < \tau_2;\ \tau_3 + \tau_4 < \tau)$.

Let us note the following characteristics of the model (the count is kept from the moment $t = 0$): $\tilde{y}(t)$ is the total quantity of output allocated for exchange for raw material; $\tilde{z}(t)$ is the total quantity of output issued by production to the warehouse, supplemented by the amount of the initial stock; $\widetilde{w}(t)$ is the total quantity of output taken out of the warehouse; $\tilde{x}(t) = \tilde{z}(t) - \tilde{y}(t) - \widetilde{w}(t)$ is the quantity of output in the warehouse at time t $(\tilde{x}(t) \geq 0)$.

These characteristics are represented by broken lines, which greatly complicates operating with them. In view of this, we shall seek expressions for other, smoother theoretical characteristics of the model under consideration: $y(t)$, $z(t)$, $w(t)$, which at the moments $t = k\tau$ $(k = 0, 1, 2, ...)$ coincide with the preceding ones. In this case the quantity $x(t) = z(t) - y(t) - w(t)$ will express the theoretical quantity of output at the disposal of production, i.e., located at the given moment in production and in the warehouse. At the moments $t = k\tau$ $(k = 0, 1, 2, ...)$, the quantity $x(t)$ expresses the real quantity of output located in the warehouse.

We distinguish two principal modes of operation of production and, for each mode, construct the corresponding characteristics.

The 1st principal mode of production occurs at nominal capacity, i.e.

$$u_1^k = \frac{Q}{\beta\tau_1}.$$

The 2nd principal mode of production occurs at a capacity below nominal, i.e.

$$u_1^k < \frac{Q}{\beta\tau_1}.$$

When studying the 1st principal mode, it is first of all necessary to take into account the strictly regulated removal from the warehouse of surplus finished output, characterized by the sequence u_2^k $(k = 1, 2, ...)$. This sequence completely determines the continuous function $\widetilde{w}(t)$, which satisfies the relations:

$$\widetilde{w}(k\tau+\tau)-\widetilde{w}(k\tau)=\tau_4 u_2^k+\tau_3 u_2^{k+1}, \qquad \widetilde{w}(0)=0 \quad (k=1,2,...).$$

Suppose that the sequence u_2^k is such that there exists an analytic function $\tilde{u}_2(t)$, defined for $t \geq 0$ and satisfying the condition $\tilde{u}_2(k\tau)=u_2^k$ for $k=1,2,....$ We shall use the expression

$$w(t+\tau)-w(t)=\tau_4\tilde{u}_2(t)+\tau_3\tilde{u}_2(t+\tau) \tag{1}$$

for the analytic function $w(t)$, defined for $t \geq 0$. At the same time, obviously, the constructed function $w(t)$ will, for $t=k\tau$ $(k=1,2,...)$, satisfy the relation $w(t)=\widetilde{w}(t)$, which indicates that at these moments $w(t)$ expresses the total quantity of output taken from the warehouse from the moment $t=0$ to the moment $t=k\tau$.

Let us find an expression for the derivative of $w(t)$ with respect to t. To this end, to expression (1) we apply the operator inverse to the finite-difference operator, and then the differentiation operator (see ([3])), i.e., the operator

$$\frac{d}{dx}\frac{1}{\Delta}=\frac{d/dx}{e^{d/dx}-E}=\sum_{\nu=0}^{\infty}\frac{B_\nu}{\nu!}\frac{d^\nu}{dx^\nu},$$

in which B_ν $(\nu=1,2,...)$ are the Bernoulli numbers.

Taking into account that in the last expression $x=\dfrac{t}{\tau}$, and denoting $\tau_3+\tau_4=\lambda$ and $u_2(t)=\dfrac{\lambda}{\tau}\,\tilde{u}_2(t)$, we obtain the expression

$$\frac{dw}{dt}=\frac{\tau_3}{\lambda}\tau\frac{du_2}{dt}+\tau\sum_{\nu=0}^{\infty}\frac{B_\nu\tau^{\nu-1}}{\nu!}\frac{d^\nu}{dt^\nu}u_2. \tag{2}$$

For the function $u_2(t)=A\beta^{t/\tau}$, where $\beta>1$, the convergence condition for the series (2) will be $\beta<e^{2\pi}\cong 530$, which indicates a very large range of possible values for β and a very high rate of convergence of the series (2) for values of β close to unity.

Since in the first basic mode operation is carried out at nominal capacity, i.e., the rate u_1^k in this case is constant and equal to $\dfrac{Q}{\beta\tau_1}$, it is expedient to take as characteristics $y(t)$ and $z(t)$, approximating $\tilde{y}(t)$ and $\tilde{z}(t)$, the linear functions

$$y(t)=\frac{Q}{\tau\beta}t+y(0), \qquad z=\frac{Q}{\tau}t+z(0),$$

which at the points $t=k\tau$ $(k=1,2,...)$ satisfy the equalities $y(t)=\tilde{y}(t)$, $z(t)=\tilde{z}(t)$.

Taking into account that $x(t) = z(t) - y(t) - w(t)$, we introduce the quantity

$$x^0(t) = x(t) + \tau\frac{\tau_3}{\lambda}u_2(t),$$

which denotes the theoretical quantity of output at the disposal of production, supplemented by the quantity of preliminary removal, i.e., of removal carried out before the end of the cycle. We obtain the differential equation

$$\frac{dx^0}{dt} = (\beta - 1)u_1 - \sum_{k=0}^{\infty}\frac{B_k\tau^k}{k!}\,\frac{du_2^k}{dt^k} \tag{3}$$

and the equality $u_1(t) = Q/\beta\tau$, characterizing the operation of production in this mode.

Considering the second basic mode, we note that in each cycle, at the moment $(k+1)\tau - \tau_2$, there is absolutely no product in the warehouse. By this moment, from the beginning of the $(k+1)$-st cycle, production will have consumed raw material in the amount $\tau_1 u_1^{k+1}$ and product will have been removed from the warehouse in the amount $\tau_4 u_2^k$. By the end of the cycle $t = (k+1)\tau$, from the moment $(k+1)\tau - \tau_2$, product from production will have arrived at the warehouse in the amount $\tau_2 u_1^{k+1} = \beta u_1^{k+1}\tau_1$, and product will have been removed from the warehouse in the amount $\tau_3 u_2^{k+1}$. Therefore, for the desired characteristics $x(t)$ and $y(t)$ at $t = k\tau$ we may write

$$y(t+\tau) = y(t) + \sigma(t),$$

where $\sigma(t) = \tau_1 u_1^{k+1}$, $x(t) = \sigma(t) + \tau_4 u_2^k$, $x(t+\tau) = \beta\sigma(\tau) - \tau_3 u_2^{k+1}$.

Taking into account the previously adopted assumption that for the sequence u_2^k $(k = 1, 2, ...)$ there exists an analytic function $u_2(t)$, defined for $t \geq 0$ and satisfying the condition

$$u_2(k\tau) = \frac{\lambda}{\tau}\,u_2^k \quad (k = 1, 2, ...),$$

which determines $w(t)$ by formula (2), after summation by the Euler-Maclaurin formula of the sequences for $x(t)$, $\sigma(t)$, and $y(t)$ at $t = k\tau$ $(k = 1, 2, ...)$, we obtain that the differential equation for x^0 has

the same form (3), while the relation for $u_1(t) = dy/dt$ takes the form

$$\tau\frac{\beta - 1}{\ln\beta}\,u_1 + \sum_{k=0}^{\infty}A_k(\beta)\,\tau^{k+1}\frac{d^k u_2}{dt^k} = x^0(t),$$

where

$$A_k(\beta) = -\sum_{n=k+1}^{\infty} C_n^k \frac{B_n}{n!} (-\ln\beta)^{n-k-1},$$

and B_n are Bernoulli numbers.

Thus, finally, we have that the production model under consideration is described by the differential equation (3) and the inequalities

$$u_1(t) \leqslant \frac{Q}{\tau\beta}, \tag{4}$$

$$\tau\frac{\beta-1}{\ln\beta}\, u_1(t) + \sum_{k=0}^{\infty} A_k(\beta)\,\tau^{k+1}\frac{d^k u_2}{dt^k} \leqslant x^0(t). \tag{5}$$

On this basis we can distinguish five operating regimes of production (so named by us conditionally):

1. The **development** regime belongs to the 2nd basic regime and is therefore described by the equality obtained from inequality (5) under the condition of no shipment, i.e. $u_2(t) \leqslant 0$.
2. The **accumulation** regime belongs to the 1st basic regime and is described by the equality obtained from inequality (4) under the condition of no shipment, i.e. $u_2(t) \leqslant 0$.
3. The **excess** regime belongs to the 1st basic regime and is described by the equality obtained from inequality (4) in the presence of shipment, i.e. $u_2(t) < 0$.
4. The **strained** regime belongs to the 2nd basic regime and is described by the equality obtained from inequality (5) in the presence of shipment, i.e. $u_2(t) < 0$.
5. The **balanced** regime belongs to the 1st basic regime, but is described at once by both equalities obtained from inequalities (4) and (5) in the presence of shipment, i.e. $u_2(t) < 0$.

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Note: Figure translations are in progress. See original paper for figures.