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Abstract

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MULTIDIMENSIONAL PROBLEMS OF THE CALCULUS OF VARIATIONS IN THE LARGE

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The principal difficulty in applying the technique of the calculus of variations in the large to multidimensional problems consists in constructing shortening deformations. In this article a new method of deformation is proposed, one that does not rely on uniqueness "in the small" : the deformations are carried out directly along curves of steepest descent for the variational functional (parabolic descent). With the aid of the deformations thus constructed, theorems on the existence of stationary solutions of variational problems are proved.*

1. **Stationary solutions.** Let the variational functional

$$J(u) = \int_\Omega F(x, u, u_x)\, dx, \qquad u\big|_S = \varphi, \tag{1}$$

be defined on functions $u(x) \in W_m^1(\Omega)$, given in a bounded domain Ω with boundary S. Suppose that

$$F(x, u, p) \in C^{2,\xi\alpha}, \qquad S \in C^{2,\xi\alpha}, \qquad \varphi \in C^{2,\xi\alpha}; \tag{2}$$

$$\nu_1(|u|)p^m \leq F(x, u, p) \leq \mu_1(|u|)(1 + p)^m; \tag{3}$$

$$\nu_2(|u|)(1 + p)^{m-2} \sum \xi_i^2 \leq F_{p_i p_j} \xi_i \xi_j \leq \mu_2(|u|)(1 + p)^{m-2} \sum \xi_i^2; \tag{4}$$

$$\left| \frac{\partial^s F(x, u, p)}{\partial p_{i_1}^{l_1} \dots \partial p_{i_n}^{l_n} \partial u^{j_0} \partial x_1^{j_1} \dots \partial x_n^{j_n}} \right| \leq \mu_3(|u|)(1 + p)^{m-(l_1+\dots+l_n)}. \tag{5}$$

Definition 1. A stationary solution of the variational problem (1) is a function $u(x_1, \dots, x_n) \in W_m^1(\Omega)$ for which the first variation

$$\delta J(u, \eta) = \int_\Omega (F_{u_{x_i}} \eta_{x_i} + F_u \eta)\, dx = 0 \tag{6}$$

for all $\eta \in \overset{\circ}{W}{}^1_m(\Omega)$. Since constrained stationary solutions, if they exist, have the same differential properties as solutions of the absolute-minimum problem (^{8}, ^{10}), in particular they satisfy the Euler equation (7):

$$\frac{\partial}{\partial x_i} F_{p_i} - F_u = 0, \tag{7}$$

the problem of **existence** is the fundamental one in the theory of stationary solutions.

2. **Parabolic descent.** Consider the boundary-value problem for the equation of parabolic type corresponding to the Euler equation:

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x_i} F_{p_i} - F_u, \qquad u\big|_S = \varphi, \qquad u\big|_{t=0} = \psi, \tag{8}$$

where $\varphi, \psi \in C^{2,\alpha}$, and for the functions φ and ψ the compatibility conditions of order zero and one hold.

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* The results of the work were reported in July 1961 at the IV All-Union Mathematical Congress in Leningra

Theorem 1. If the boundary-value problem (8) has, in the cylinder $Q = \Omega \times [0, \infty)$, a solution $u(x,t)$, then for it

$$\int_t^T dt \int_\Omega \left(\frac{\partial u}{\partial t}\right)^2 dx = J(u(x,t)) - J(u, x, T), \tag{9}$$

i.e., the parabolic trajectory $u(x,t)$ is a curve of steepest descent for the variational functional (1).

Impose on $F(x,u,p)$ the additional condition:

$$|F_u| \le c\left\{|u|^\alpha \xi(x) + |u|^\beta \zeta(x) p^{m-\varepsilon}\right\} \tag{10}$$

under: 1) $\alpha = m\frac{n+q}{n} - 1 - \delta_\alpha q$, where $q \ge \frac{nm}{n-m}$, $\delta_\alpha > 0$; 2) $\xi(x) \in \mathcal{L}_{r_\alpha}(\Omega)$, where $r_\alpha > \max\left(\frac{1}{\delta_\alpha}, \frac{n}{m}\right)$; 3) $\frac{n}{n+q} < \varepsilon < m$; 4) $\beta = \varepsilon\frac{n+q}{n} - 1 - \delta_\beta q$, where $\delta_\beta > 0$; 5) $\zeta(x) \in \mathcal{L}_{r_\beta}(\Omega)$, where $r_\beta > \max\left\{\frac{m}{\varepsilon\delta_\beta}, \frac{n}{\varepsilon}\right\}$ (we note that in this condition the most stringent restriction is the exponent $m - \varepsilon$).

The additional condition (10) was formulated in [11], where, under this condition, boundedness was proved for stationary solutions belonging to $\mathcal{L}_q(\Omega) \cap W^1_m(\Omega)$ of the variational problem (1).

Theorem 2. If for the function $F(x,u,p)$ conditions (2)–(5) are fulfilled with $m=2$ and the additional condition (10) is fulfilled, then, under the condition $\|u(x,t)\|_{\mathcal{L}_q(\Omega)} \leq M_0$, the boundary-value problem (8) has in the cylinder $Q_T = \Omega \times [0,T]$ a unique solution $u(x,t) \in C^{2,\alpha}$.

In [12] the existence of solutions and a priori estimates were obtained under conditions (2)–(5) and an additional condition close to (10); in [13]—under conditions (2)–(5) ($m=2$). Theorem 2 follows from the results of these works.

Theorem 3. If the function $F(x,u,p)$ satisfies conditions (2)–(5), (10) and, for the solution $u(x,t)$ of the boundary-value problem (8) in the cylinder $Q = \Omega \times [0,\infty)$, uniformly in t

$$\|u(x,t)\|_{\mathcal{L}_q(\Omega)} < \dot{M}, \tag{11}$$

then, first, the solution $u(x,t)$ is uniformly bounded, $|u(x,t)| < M_1$, and, second, there exists a function $u_0(x)$ such that

$$\lim_{t\to\infty} \|u(x,t) - u_0(x)\|_{C^{1,\alpha}(\Omega)} = 0, \qquad \lim_{t\to\infty} |J(u(x,t)) - J(u_0(x))| = 0,$$

i.e., the solution $u(x,t)$ as $t \to \infty$ stabilizes to $u_0(x)$.

The proof is analogous to [11-13].

3. The principle of a critical point. Let a general variational functional $J(u)$ be defined in a space $\mathcal{L}$. Denote by J_c the domain of smaller values $\{J(u) \leq c\}$.

Definition 2. A contracting deformation in the domain J_b will mean a continuous mapping $D_t(u) : J_b \times [0,\infty) \to J_b$ such that: 1) for any $u_0 \in J_b$ there exists a limit $\lim_{t\to\infty} D_t(u_0) = u_\infty$, and moreover $\lim_{t\to\infty} J(D_t(u_0)) = J(u_\infty)$; 2) for $t_0 < T$ we have $J(D_{t_0}(u)) \geq J(D_T(u))$, with equality possible if and only if $D_t(u)$ is a stationary solution (for $t \geq t_0$).

We shall call the number

$$q(z_k) = \lim_{t\to\infty} \max_{u \in z_k} J(D_t(u)) \tag{12}$$

the **critical value of the relative singular cycle** z_k of the domain J_b modulo the domain of smaller values J_a ($a < b$), and we shall call the number

$$Q\{z_k\} = \inf_{z_k \in \{z_k\}} q(z_k). \tag{13}$$

the **critical value of the homology class** $\{z_k\}$ containing z_k.

If $Q\{z_k\} - a > 0$, then the homology class $\{z_k\}$ is separated by a positive barrier from the domain of smaller values J_a. A cycle $z_k \in \{z_k\}$ will be called fundamental if $q(z_k) = Q\{z_k\}$ and, for all functions $u \in z_k$, the uniform estimate (4) is satisfied for $0 \leq t < \infty$.

Theorem 4 (principle of the critical point). *If the homology class $\{z_k\}$ is separated by a positive barrier from the domain J_a and in the class $\{z_k\}$ there is a fundamental cycle, then the variational problem (1), under conditions (2)–(5) and (10), has a stationary solution $u_0(x)$ for which*

$$J(u_0(x)) = Q\{z_k\}.$$

For the proof it is enough to define the contracting deformation $D_t(v) = u(x,t)$, where $u(x,t)$ is the solution of problem (8) with initial function $u(x,0) = v(x)$, and then apply Theorem 3.

The definitions and Theorem 4 can be formulated analogously for homotopy classes.

4. Existence of stationary solutions. The barrier theorem

Theorem 5. *Let the variational functional (1) satisfy the natural conditions (2)–(5) with order of growth $m = 2$ and the additional condition (10). If every homology class of the groups $H_k(C^{2,\alpha}(\Omega), J_a)$ is separated by a positive barrier from the domain J_a and contains a fundamental cycle, then the algebraic number of stationary solutions of the variational problem is not less than the number of all independent cycles in the domain of smaller values J_a.*

Indeed, from the exact sequence

$$\cdots \to H_k(J_a) \to H_k(C^{2,\alpha}(\Omega)) \to H_k(C^{2,\alpha}(\Omega), J_a) \to H_{k-1}(J_a) \to \cdots \tag{14}$$

it follows that the groups $H_k(C^{2,\alpha}(\Omega), J_a)$ and $H_{k-1}(J_a)$ are isomorphic.

It now suffices to apply the critical-point principle to each homology class of the group $H_k(C^{2,\alpha}(\Omega), J_a)$.

Corollary. *If a variational problem has two isolated solutions with relative minimum, then there exists one more stationary solution.*

This proposition is analogous to the Morse–Courant barrier theorem [9] on minimal surfaces.

Remark. Theorems 4 and 5 show that, in those cases where the existence and stabilization theorems for parabolic trajectories can be proved, one can obtain existence theorems for stationary solutions of the corresponding variational problems.

5. One-dimensional problems

Deformations along trajectories of parabolic descent can be applied to one-dimensional variational problems. Thus, for the problem on geodesics on a Riemannian manifold the system of equations of parabolic type has the form:

$$\frac{dx^j}{dt} = \frac{d^2x^j}{ds^2} + \Gamma^j_{kl}(x)x^k_s x^l_s \quad (j = 1, 2, \ldots, n). \tag{15}$$

For such systems the necessary existence theorems have been proved, and therefore the fundamental estimates in one-dimensional problems of the calculus of variations in the large can be proved by using parabolic descent.

6. Harmonic and minimal surfaces on Riemannian manifolds

Definition 3. By a closed surface of a given topological type on a Riemannian manifold $\mathfrak{M}^n$ is meant a continuous mapping of class W^1_2 of the canonical polygon $K \overset{x}{\to} \mathfrak{M}^n$ which coincides on the corresponding "identification" intervals.

The first variation of the Dirichlet integral

$$D(x(u,v)) = \frac{1}{2}\int_K [(x_u, x_u) + (x_v, x_v)]\, du\, dv \tag{16}$$

has the form

$$\delta D(x, \vec{\eta}) = \int_K \left[\left(\frac{D\vec{\eta}}{du}, x_u\right) + \left(\frac{D\vec{\eta}}{dv}, x_v\right)\right] du\, dv. \tag{17}$$

Here $\vec{\eta} = \vec{\eta}[x(u,v)]$ is a vector field on the surface $x(u,v)$. If $\delta D(x, \vec{\eta}) = 0$ for every vector field of class W^1_2, then the closed surface is called **harmonic**.

The system of equations of parabolic descent for the Dirichlet integral has the form

$$\frac{\partial x^j}{\partial t} = \Delta x^j + \Gamma^j_{kl}(x)(x^k_u x^l_u + x^k_v x^l_v). \tag{18}$$

But for such systems the existence of solutions has not yet been proved; therefore contracting deformations have been constructed only for manifolds of nonpositive curvature.

Theorem 6. Let a harmonic surface $x(u,v) \in C^{2,\alpha}$ be given, and let $y(u,v)$ be an arbitrary surface of class $C^{2,\alpha}$ with the same boundary condition $x|_\Gamma = y|_\Gamma$, where $\max_{(u,v)\in K} \|y(u,v) - x(u,v)\| < \rho_0$, the elementary length on the manifold. Join the points of the surfaces $x(u,v)$ and $y(u,v)$ by the shortest curve $z(t; u, v)$ with reduced parameter t $(0 \le t \le 1)$. Then

$$D(y(u,v))-D(x(u,v)) = \int_0^1 d\tau \int_0^\tau dt \int_K \Big[\Big(\frac{Dz_u}{dt}, \frac{Dz_u}{dt} \Big) + \Big(\frac{Dz_v}{dt}, \frac{Dz_v}{dt} \Big) - k(z_u, z_t) s^2(z_u, z_t) - k(z_v, z_t) s^2(z_v, z_t \tag{19}$$

Here $k(\vec{\xi}, \vec{\eta})$ is the Riemannian curvature at the point $z(t; u, v)$ in the direction of the bivector constructed on the vectors $\vec{\xi}, \vec{\eta}$; $s(\vec{\xi}, \vec{\eta})$ is the modulus of this bivector.

From (19) it follows that

Theorem 7. If the Riemannian curvature of the manifold $\mathfrak{M}^n$ is negative, then $D(y(u,v)) - D(x(u,v)) > 0$, and, consequently, the harmonic surface is unique in its ρ_0-neighborhood.

Remark. In [16] it was asserted that Theorem 7 is true for surfaces of disk type under arbitrary curvature, but examples show that this is not so.

Theorem 8. The spaces of closed surfaces of fixed topological type on a compact Riemannian manifold $\mathfrak{M}^n$ of negative curvature consist of homotopically trivial components. In each component there exists a unique harmonic and a unique minimal closed surface. There are no stationary surfaces on a manifold of nonpositive Riemannian curvature.

Remark 1. On a manifold of nonpositive curvature there may exist blocks of harmonic and minimal surfaces.

Remark 2. An analogous theorem is valid for spaces of k-dimensional surfaces.

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