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Abstract

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EXISTENCE AND ASYMPTOTICS OF SOLUTIONS OF THE "SOLITARY WAVE" TYPE FOR DIFFERENTIAL EQUATIONS IN A BANACH SPACE

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The problem considered in this note describes a phenomenon well known in mechanics under the name "solitary wave" and first studied by M. A. Lavrent' ev ([1]) (see also ([2,3])). Recently A. M. Ter-Krikorov and the author showed ([4]) that this phenomenon occurs for a certain class of quasilinear elliptic equations in a rectilinear unbounded strip. Below, the validity of analogous results is established for a second-order differential equation in a Banach space. Our investigation makes essential use of the theory of semigroups.

Lemma 1. Let an operator B act in a Banach space E, where:

1) B is a closed linear unbounded operator with domain dense in E;
2) $-B$ is the infinitesimal generator of a strongly continuous semigroup $\exp(-\xi B)$, $\xi \geqslant 0$, such that

$$\|\exp\{-\xi B\}\| \leqslant \exp\{-\xi\beta\}, \qquad \beta = \text{const} > 0.$$

Define $B^{1/2}$ to be a closed linear unbounded operator with domain dense in E, as the inverse of the operator (see ([5]))

$$B^{-1/2} = \frac{1}{\sqrt{\pi}} \int_0^{+\infty} \exp\{-\eta B\}\eta^{-1/2}\, d\eta.$$

Then the operator $-B^{1/2}$ is the infinitesimal generator of the strongly continuous semigroup $\exp\{-\xi B^{1/2}\}$, $\xi \geqslant 0$, and moreover

$$\|\exp\{-\xi B^{1/2}\}\| \leqslant \exp\{-\xi\sqrt{\beta}\}.$$

Proof. A related result is available in ([6]). Our proof is based on the following integral representation, valid for $\lambda \in (-\sqrt{\beta}, +\infty)$:

$$(B^{1/2} + \lambda I)^{-1} = \frac{1}{\sqrt{\pi}} \int_0^{+\infty} \exp\{-\eta B\} \psi(\lambda\sqrt{\eta}) \eta^{-1/2}\, d\eta,$$

where

$$\psi(x) = 1 - 2xe^{x^2} \int_x^{+\infty} e^{-t^2}\, dt > 0.$$

From this representation one derives an estimate for the norm of the operator $(B^{1/2} + \lambda I)^{-1}$, which makes it possible to apply the Hille–Yosida–Phillips theorem ([7]) and obtain the assertion of the lemma.

We now introduce, for $\delta > 0$, E_δ—the Banach space of abstract even functions $f(\xi)$, continuous on $(-\infty, +\infty)$, with norm

$$|f(\xi)|_\delta = \sup_{(-\infty,+\infty)} \|e^{\delta|\xi|} f(\xi)\|.$$

Lemma 2. Suppose that the conditions of Lemma 1 are fulfilled for the operator B, and suppose $h(\xi) \in E_\delta,\ h'(\xi) \in E_\delta$, where $0 < \delta < \sqrt{\beta}$. Then the boundary-value problem

$$-\frac{d^2 z}{d\xi^2} + Bz = h(\xi), \qquad \lim_{\xi \to \pm\infty} z = 0$$

has a solution $z = z(\xi),\ z^{(k)}(\xi) \in E_\delta,\ k = 0, 1, 2$. This solution is given by the explicit formula

$$z(\xi) = \frac{1}{2} \int_{-\infty}^{+\infty} \exp\{-|\xi - \eta| B^{1/2}\} B^{-1/2} h(\eta)\, d\eta.$$

The validity of Lemma 2 is verified by direct computations. We now pass to the problem of a solitary wave; namely, we consider in the Banach space E the following nonlinear boundary-value problem:

$$-\frac{d^2 y}{d\eta^2} + Ay = F(\lambda, y), \qquad -\infty < \eta < +\infty, \qquad \lim_{\eta \to \pm\infty} y(\eta) = 0. \tag{1}$$

Here λ is a real parameter; A is a closed linear unbounded operator with dense domain of definition in E; $F(\lambda, y)$ is a nonlinear operator acting in E, analytic in the Fréchet sense with respect to λ, y in some neighborhood of the point $y = 0$ for all λ, and $F(\lambda, 0) = 0$. Problem (1) always has the trivial solution. Our aim is to give conditions sufficient for a small nontrivial solution to branch off from the trivial solution for some $\lambda = \lambda_0$.

Suppose there exists λ_0 such that the operator $B = A - \partial F(\lambda_0, 0)/\partial y$ satisfies the following conditions:

1) Zero is a simple isolated eigenvalue of the operator B, with corresponding null element φ.
2) For solvability of the equation $By = h$ it is necessary and sufficient that $\psi(h) = 0$, where ψ is some linear functional in E; moreover φ and ψ can be normalized so that $\psi(\varphi) = -1$.
3) Decompose E into the direct sum $E = E^1 \dot{+} E^{\infty-1}$, where E^1 is the null subspace of the operator B, and $E^{\infty-1}$ is its range. In $E^{\infty-1}$ the operator B is invertible. We require that, for B considered on $E^{\infty-1}$, the conditions of Lemma 1 be fulfilled. Put now $\lambda = \lambda_0 + \varepsilon$ and write $F(\lambda, y)$ in the form

$$F(\lambda_0 + \varepsilon, y) = F_0 y + \sum_{j=2}^{\infty} F_{0j} y^j + \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} F_{ij} \varepsilon^i y^j.$$

The subsequent exposition uses the definition of a generalized Jordan chain, which generalizes the previously given definitions (see ([4],[8],[9])).

Let the operator $\Phi(y)$ be analytic in the Fréchet sense in some neighborhood of zero, with $\Phi^{(i)}(0) = 0,\ i = 0, 1, \dots, m-1$, but $\Phi^{(m)}(0) \neq 0$. Let $\Phi_i(y_1, y_2, \dots, y_i)$ be the coefficients in the formal expansion

$$\Phi\left(\sum_{k=1}^{\infty} y_k \varepsilon^k\right) = \sum_{i=1}^{\infty} \Phi_i(y_1, y_2, \dots, y_i) \varepsilon^{m+i-1}.$$

Definition. We shall say that the operator B has, relative to the operator $\Phi(y)$, a Jordan chain of length ρ, if there exist ρ linearly independent elements

$\varphi_1, \varphi_2, \dots, \varphi_p$, satisfying the relations $B\varphi_1 = 0$, $B\varphi_k = \Phi_{k-1}(\varphi_1, \dots, \varphi_{k-1})$, $k = 2, \dots, p$, and moreover

$$\psi(\Phi_p(\varphi_1, \dots, \varphi_p)) \neq 0.$$

Theorem 1. Suppose that the operator B has, relative to the operator

$$\Phi(y) = \sum_{j=2}^{\infty} F_{0j} y^j,$$

a Jordan chain of length p, and suppose that $\alpha = \psi(F_{11}\varphi) > 0$. Put $a_p = \psi(\Phi_p(\varphi_1, \dots, \varphi_p))$.

Then, if $p(m-1)$ is odd, for all λ sufficiently close to λ_0, and if $p(m-1)$ is even, for all λ sufficiently close to λ_0 and satisfying the inequality $a_p(\lambda_0 - \lambda) > 0$, there exists a nontrivial solution of problem (1) in the space E_δ, where $0 < \delta < \min(\sqrt{\beta},\, 2(m-1)\sqrt{\alpha})$. The following asymptotics hold ($\lambda > \lambda_0,\ N > p$):

$$y(\eta, \varepsilon) = \sum_{i=1}^{N} y_i(\eta\sqrt{\lambda - \lambda_0})(\lambda - \lambda_0)^{[1+(i-1)(m-1)]/p(m-1)} +$$

$$+ O((\lambda - \lambda_0)^{[1+N(m-1)]/p(m-1)}).$$

Remark. Together with $y(\eta)$, $y(\eta + c)$ will be a solution of problem (1) for any c, since (1) is invariant with respect to shifts in η. The asymptotics is constructed in the same way as in (4); its principal term has the form

$$\left[\frac{\alpha(\lambda_0 - \lambda)}{a_p}\right]^{1/p(m-1)} \left[\operatorname{ch} \frac{1}{2} p(m-1)\sqrt{\alpha|\lambda_0 - \lambda|}\,\eta\right]^{-2/p(m-1)} \varphi.$$

For the proof of the existence of a solution, Lemma 2 is used.

The solitary wave found in Theorem 1 has exponential decay at infinity. We give conditions under which the decay is of power type.

Let L_{ij} be the coefficients of the branching equation composed for finding small solutions of the equation $Ay = F(\lambda_0 + \varepsilon, y)$ (see $(10, 11)$). Introduce E_*, the Banach space of abstract even functions $f(\xi)$, continuous on $(-\infty, +\infty)$, with norm

$$|f(\xi)|_* = \sup_{(-\infty,+\infty)} \|f(\xi)\omega(\xi)\|,$$

where

$$\omega(\xi) = (1 + \xi^2)^{3/2} \ln(e + |\xi|).$$

Theorem 2. Suppose $L_{11} = L_{12} = L_{20} = 0$, $L_{21} \neq 0$, $L_{30} < 0$. Then, for all λ sufficiently close to λ_0, there exists a nontrivial solution of problem (1) in the space E_*. The asymptotics holds

$$y(\eta, \varepsilon) = \sum_{i=1}^{N} y_i(\eta(\lambda - \lambda_0))(\lambda - \lambda_0)^i + O((\lambda - \lambda_0)^{N+1}).$$

The scheme of the proof is the same. We note only that in Lemma 2 the space E_δ can be replaced by the space E_*, and lemmas analogous to Lemmas 1 and 2 from (4) can be proved. The principal term of the asymptotics here will be

$$\frac{6(\lambda-\lambda_0)}{L_{21}}\left[\eta^2(\lambda-\lambda_0)^2-\frac{9}{2}L_{30}\right]^{-1}\varphi.$$

Example. $E = L_q(\Omega)$, where Ω is a simply connected bounded domain in R^n with boundary Γ,

$$Ay \equiv -\sum_{i,j=1}^{n} a_{ij}(\xi)\frac{\partial^2 y}{\partial\xi_i\partial\xi_j}+\sum_{i=1}^{n} a_i(\xi)\frac{\partial y}{\partial \xi_i}+a(\xi)y,$$

a_{ij}, a_i, a are sufficiently smooth, $\xi = (\xi_1, \dots, \xi_n)$, and

$$\sum_{i,j=1}^{n} a_{ij}(\xi)\gamma_i\gamma_j \geq k\sum_{i=1}^{n}\gamma_i^2, \qquad k = \text{const} > 0.$$

The domain of definition of A consists of functions belonging to $W_q^{(2)}(\Omega)$ for which $y|_\Gamma = 0$. Let

$$F(\lambda, y) = \sum_{i=1}^{n} F_i(\lambda,\xi)y^i,$$

where $F_i(\lambda, \xi)$ are sufficiently smooth. In the infinite cylinder $\Omega \times (-\infty, +\infty)$ consider the elliptic equation

$$\frac{\partial^2 y}{\partial\eta^2}+\sum_{i,j=1}^{n} a_{ij}(\xi)\frac{\partial^2 y}{\partial\xi_i\partial\xi_j}-\sum_{i=1}^{n} a_i(\xi)\frac{\partial y}{\partial\xi_i}-a(\xi)y+F(\lambda,\xi,y)=0$$

with boundary conditions

$$y\big|_{\Gamma\times(-\infty,+\infty)} = 0, \qquad \lim_{\eta\to\pm\infty} y = 0.$$

For this problem Theorems 1 and 2 can be rephrased. The conditions of Lemma 1 are verified in ([12]). One can also indicate conditions under which the solution found will be classical.

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Note: Figure translations are in progress. See original paper for figures.