A BOUNDARY-VALUE PROBLEM OF CARLEMAN-PROBLEM TYPE FOR A MULTIPLY CONNECTED DOMAIN

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Abstract

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E. I. ZVEROVICH

A BOUNDARY-VALUE PROBLEM OF CARLEMAN-PROBLEM TYPE FOR A MULTIPLY CONNECTED DOMAIN

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Let D be a finite $(m+1)$-connected domain with Lyapunov contour L, consisting of the curve L_0, enclosing all the remaining curves $L_1, \dots, L_m$. By $D_0^-, \dots, D_m^-$ we denote the complement of $D+L$ to the plane. By $\omega(t, L_k)$ $(k = 0, \dots, m)$ we denote the harmonic measures (^{1}) of the boundary contours L_k with respect to the domain D. Let

$$a(t) = \sum_{k=0}^{m} \omega(t, L_k)\alpha_k(t),$$

where $\alpha_k(t)$ is an orientation-preserving homeomorphism of the contour L_k onto itself, satisfying the conditions: $\alpha_k[\alpha_k(t)] \equiv t$ (the Carleman condition) and $\alpha_k'(t)$ is H-continuous. On L two H-continuous functions are given: $G(t) \neq 0$ and $g(t)$.

The problem is posed:

To find a function $\Phi(z)$, single-valued and analytic in D, H-continuous in $\overline{D}$, satisfying one of the conditions:

$$\Phi^+[\alpha(t)] = G(t)\overline{\Phi^+(t)} \quad \text{on } L, \tag{1}$$

$$\Phi^+[\alpha(t)] = G(t)\overline{\Phi^+(t)} + g(t) \quad \text{on } L. \tag{2}$$

Of interest is the case when the conditions (^{2},^{3}) are satisfied:

$$G[\alpha(t)]\overline{G(t)} \equiv 1, \qquad G[\alpha(t)]\overline{g(t)} + g[\alpha(t)] \equiv 0. \tag{3}$$

The case when D is a simply connected domain was studied in (^{3}). In the case when $\alpha(t) \equiv t$ and $\varkappa_k = \operatorname{ind} G(t)|_{L_k}$ $(k = 0, \dots, m)$ are even numbers, we have the Hilbert problem ($^{4-6}$).

§ 1. **Lemma 1.** *A function $\Phi(z)$, single-valued and analytic in D, H-continuous in $\overline{D}$, can be represented in the form*

$$\Phi(z) = \frac{1}{2\pi i} \int \frac{\varphi[\alpha(\tau)]}{\tau - z}\, d\tau + i \int \omega(\tau, L_m)\varphi(\tau)\, [d\sigma_\tau + d\sigma_{\alpha(\tau)}], \tag{4}$$

where $d\sigma = d\sigma_\tau$ *and* $d\sigma_{\alpha(\tau)}$ *are elements of the arc* L*, computed respectively at the points* τ *and* $\alpha(\tau)$*;* $\varphi(\tau)$ *is an* H*-continuous function satisfying the condition*

$$\varphi(\tau) + \varphi[\alpha(\tau)] \equiv 0,$$

defined up to an additive term of the form

$$i \sum_{k=0}^{m-1} \mu_k \omega(\tau, L_k),$$

where μ_k *are arbitrary real constants.*

Lemma 2. *The general solution of the problem*

$$\Phi^+[\alpha(t)] = \overline{\Phi^+(t)} \tag{5}$$

is an arbitrary real constant.

This is proved on the basis of representation (4).

In contrast to the case of a simply connected domain, problem (2) with $G(t) \equiv 1$ is, generally speaking, unsolvable. In view of this we shall consider a problem analogous to the modified Dirichlet problem (^{7}).

* Here and below all integrals are taken over L.

Lemma 3. The nonhomogeneous problem

$$\Phi^+[\alpha(t)] = \overline{\Phi^+(t)} + g(t) + i \sum_{k=1}^{m} c_k \omega(t, L_k) \tag{6}$$

(where it is required to find real constants $c_1, \dots, c_m$ so that a solution of problem (6) exists, and then the solution itself) is always solvable. The constants $c_1, \dots, c_m$ are determined uniquely.

For the proof it is enough to consider the case $g(t) \equiv 0$. Using Lemma 1, we reduce problem (6) to the integral equation

$$\varphi(t) + \frac{1}{2\pi i} \int \left[\frac{\alpha'(\tau)}{\alpha(\tau) - \alpha(t)} - \frac{\overline{\tau}^2}{\tau - t} \right] \varphi(\tau)\, d\tau = i \sum_{k=1}^{m} c_k \omega(t, L_k). \tag{7}$$

It is immediate to verify that the functions $i\omega(t, L_k)$ $(k = 1, \dots, m)$ are eigenfunctions of equation (7). Therefore the left- and right-hand sides of equality (7) are mutually orthogonal, whence it follows that $c_1 = \dots = c_m = 0$.

Lemma 4. The problem

$$\Phi^+[\alpha(t)] = \exp\left[2\pi i \sum_{k=1}^{m} c_k \omega(t, L_k)\right] \overline{\Phi^+(t)} \tag{8}$$

has as its general solution an arbitrary real constant if $c_k \equiv 0 \pmod 1$ $(k = 1, \dots, m)$, and in all other cases is unsolvable.

If all c' s are rational numbers, then the assertion follows from the fact that a sufficiently high power of equality (8) gives a condition of the form (5). In the remaining cases problem (8) is reduced to an integral equation, and the continuous dependence of its resolvent on the kernel is used.

We shall call the index of problem (1) the quantity $\varkappa = \sum_{k=0}^{m} \varkappa_k$, where $\varkappa_k = \operatorname{ind} G(t)\big|_{L_k}$ $(k = 0, \dots, m)$.

Theorem 1. If $\varkappa < 0$, problem (1) is unsolvable $(l = 0)$.

The proof is carried out for the case when all $\varkappa_k$ are even numbers, $\varkappa_k = 2\varkappa'_k$. Using the preceding lemmas, we bring condition (1) to the form

$$\frac{\Phi^+[\alpha(t)] \prod_{k=1}^{m} [\alpha(t) - z_k]^{\varkappa'_k}}{\chi^+[\alpha(t)]\, [\alpha(t)]^{\varkappa/2}} = \exp\left[2\pi i \sum_{k=1}^{m} c_k \omega(t, L_k)\right] \left\{ \overline{\frac{\Phi^+(t) \prod_{k=1}^{m} (t - z_k)^{\varkappa'_k}}{\chi^+(t)\, t^{\varkappa/2}}} \right\}, \tag{9}$$

where c_k are certain constants, $z_k \in D_k$ $(k = 1, \dots, m)$; $\chi(z)$ is a single-valued analytic function, nowhere vanishing. For $\varkappa < 0$, Lemma 4 is applicable to equality (9), whence $\Phi(z) \equiv 0$. The case when not all $\varkappa_k$ are even reduces to this special case.

§ 2. Consider the case $\varkappa \geq 0$. A method is applied analogous to that used by I. N. Vekua (^{4,5}) for the case when $\alpha(t) \equiv t$ and all $\varkappa_k$ are even. The result obtained here generalizes the known results of I. N. Vekua and B. V. Boyarskii (^{4}). Using Lemma 1, we reduce problem (1) to the integral equation

$$[1 + G(t)]\varphi(t) + \frac{1}{\pi i} \int \left[\frac{\alpha'(\tau)}{\alpha(\tau) - \alpha(t)} - \frac{G(t)\bar{\tau}^2}{\tau - t} \right] \varphi(\tau)\, d\tau +$$
$$+ i[1 - G(t)] \int \omega(\tau, L_m) \varphi(\tau)\, [d\sigma + d\sigma_{\alpha(\tau)}] = 0. \tag{10}$$

Along with $\varphi(t)$, the function $\overline{\varphi[\alpha(t)]}$ satisfies equation (10). Equation (10) behaves analogously to the real integral equation (7); namely, its fundamental system of solutions $\varphi_1(t), \dots, \varphi_\nu(t)$ can be chosen so that

$$\varphi_j(t) + \varphi_j[\alpha(t)] \equiv 0, \quad j = 1, \dots, \nu.$$

To the solutions $i\omega(t, L_1), \dots, i\omega(t, L_{m-1})$ of equation (10) there corresponds the trivial solution of problem (1), and conversely. Hence it follows that $l = \nu - m + 1$. It can be shown that the equation adjoint to (10) has the form

$$[1 + G(t)]\psi(t) - \frac{1}{\pi i} \int \left[\frac{\alpha'(t)}{\alpha(\tau) - \alpha(t)} - \frac{G(\tau)\bar{\tau}^2}{\bar{\tau} - \bar{t}} \right] \psi(\tau)\, d\tau = 0. \tag{11}$$

Let ν' be the number of solutions of equation (11). Obviously, $\nu - \nu' = \varkappa$. Equation (11) is also analogous to a real one. All its solutions can be chosen so that they satisfy the condition

$$\psi(t) = \overline{G[\alpha(t)]} \times \psi[\alpha(t)]\, \alpha'(t) t^2.$$

If $\psi(t)$ is a solution of equation (11), then, associating equation (11) with Carleman-type problems for the domains $D_0^-, \dots, D_m^-$, one can show that the function $\psi(t) G(t) t^2 = \chi^+(t)$ is the boundary value of a single-valued analytic function. For $\chi^+(z)$ we have:

$$\chi^+[\alpha(t)] = \frac{\bar{t}^2}{\overline{G(t)}\, \alpha'(t)} \overline{\chi^+(t)}. \tag{12}$$

We shall call problem (12) the adjoint of problem (1). The number of solutions is $l' = \nu'$. Its index is $\varkappa' = -\varkappa + 2(m-1)$.

Theorem 2. *If* $\varkappa > 2(m-1)$*, then problem (1) has* $l = \varkappa - m + 1$ *solutions.*

For the proof, we note that when $\varkappa > 2(m-1)$, $\varkappa' < 0$, and therefore $l' = 0$. But $l' = \nu'$, hence $l = \varkappa - m + 1$.

Theorem 3. *In the case* $0 \leqslant \varkappa \leqslant 2(m-1)$ *(the special case), for* l *the sharp estimate holds*

$$\max\{0, \varkappa - m + 1\} \leqslant l \leqslant \left[\frac{\varkappa}{2}\right] + 1.$$

The proof is analogous to that used by B. V. Boyarskii for the case of the Hilbert problem.

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Rostov-on-Don
State University

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Note: Figure translations are in progress. See original paper for figures.