A. I. PRILEPKO

1964

SovietRxiv

Abstract

Full Text

A. I. PRILEPKO

ON AN INVERSE PROBLEM OF THE META-HARMONIC POTENTIAL

(Presented by Academician M. A. Lavrent' ev on VIII 19, 1963)

1°. In the present article a solution is given of the inverse problem of the metaharmonic potential in the following formulation.

One seeks a body T_1, if its exterior metaharmonic potential V_1 is known, close in the sense of a certain functional metric to the exterior metaharmonic potential V of a given body T. It is assumed that the body T is star-shaped with respect to some interior point, and that the boundary S of the body T is such that the functions of its parametric representation are twice differentiable and their second derivatives satisfy the Hölder condition with exponent $\lambda < 1$ (a surface of class $A^{(2,\lambda)}$). For the Newtonian potential, a problem of this kind was completely solved by V. K. Ivanov (^{1}) (under the assumption that the body T is a sphere, a similar problem was studied earlier by L. N. Sretenskii (^{2})).

2°. Let x be a point of the surface S with radius vector

$$\mathbf{R}(x) = \mathbf{R}_x(\xi, \eta).$$

Define the point $y \in E^3$ by the radius vector

$$\mathbf{R}_y = \mathbf{R}_x + \nu \mathbf{n}_x,$$

where $\mathbf{n}_x$ is the exterior normal to the surface S at the point x; ν is a given number. The triple (ξ, η, ν) may be regarded as curvilinear coordinates of the point y. It is known that, for sufficiently small ν ($|\nu| \leqslant \varepsilon_0$), distinct points y correspond to distinct triples of numbers, and conversely.

Denote by R_2 the set of functions

$$\zeta(x) \equiv \zeta(\xi, \eta),$$

defined on the surface S and belonging to the class $C^{(1,\lambda)}$ (the first derivatives of the function $\zeta(x)$ satisfy the Hölder condition with exponent $0 < \lambda < 1$). On the set R_2 introduce a norm, taking it equal to the largest of the numbers

$$4 \max |\zeta(x)|, \quad 4 \max |\zeta_\xi(x)|, \quad 4 \max |\zeta_\eta(x)|,$$

$$4 \sup \frac{|\zeta_\xi(y) - \zeta_\xi(x)|}{|y-x|^\lambda}, \quad 4 \sup \frac{|\zeta_\eta(y) - \zeta_\eta(x)|}{|y-x|^\lambda}.$$

The space with this norm is a Banach ring.

The function

$$V(x) = \int_T \frac{e^{-\varkappa r_{xy}}}{r_{xy}}\, dy$$

we shall call the **metaharmonic potential of the body T of unit density**, where $\varkappa = \text{const} > 0$, $r_{xy} = |y-x|$ is the distance between the points y and x. If the surface $S \in A^{(2,\lambda)}$, then the boundary values outside of $\partial V/\partial \nu$, $\partial^2 V/\partial \nu^2$ belong to the space R_2.

Consider a body T, bounded by a surface $S \in A^{(2,\lambda)}$ and star-shaped with respect to some interior point. Suppose the metahar-

metaharmonic potential $V(x)$ of the body T of unit density. Moreover, suppose that in the domain exterior to the surface S there is defined a metaharmonic function V_1 (i.e., a regular solution of the equation $\Delta V_1 - \varkappa^2 V_1 = 0$), which at infinity behaves like a metaharmonic potential.

We shall additionally assume that:

1) V_1 admits metaharmonic continuation through S into the body T for some positive distance d;
2) each of the quantities

$$\|V_1 - V\|, \quad \left\|\frac{\partial V_1}{\partial \nu} - \frac{\partial V}{\partial \nu}\right\|, \quad \left\|\frac{\partial^2 V_1}{\partial \nu^2} - \frac{\partial^2 V}{\partial \nu^2}\right\|$$

does not exceed ωC, where $C = C(T)$, $0 < \omega < d$, $\omega = \omega(T, V, V_1, \varepsilon_0)$.

Let $\{S_1\}$ be a family of surfaces whose equation in the curvilinear coordinate system is given in the form

$$\{\nu = \zeta(x)\}, \qquad |\nu| \leqslant \varepsilon_0, \qquad \zeta \in C^{(1,\lambda)}.$$

Under these conditions on the body T, the surfaces S and S_1, and the functions V and V_1, the following holds.

Theorem (main). *There exists, and moreover is unique, a surface S_1, bounding a body T_1, satisfying the condition $\|\zeta\| < d$, such that the exterior metaharmonic potential of the body T_1 of unit density is equal to the prescribed metaharmonic function V_1 in the domain exterior to the surface S_1.*

3°. The function determining the boundary of the sought body is a solution of a nonlinear integro-differential equation. The derivation of this equation is based on ideas of V. K. Ivanov (^{1}) and L. Lichtenstein (^{3,4}).

Let the equation of the sought surface S_1 in curvilinear coordinates have the form

$$\nu = \zeta(\xi, \eta).$$

Introduce between S and S_1 a one-parameter family of surfaces S_t, defined by the equation

$$\nu = t\zeta(\xi, \eta) \qquad (0 \le t \le 1).$$

Denote by T_t the domain bounded by the surface S_t, and introduce the points $x(\xi, \eta, 0)$, $x_t(\xi, \eta, t\zeta)$, $z_t(\xi, \eta, t\zeta + \varepsilon)$ ($\varepsilon > 0$, curvilinear coordinates).

Let $\mathbf{r}_t = \mathbf{R}(y) - \mathbf{R}(x_t)$, where $\mathbf{R}(x_t) = \mathbf{R}(x) + t\zeta\mathbf{n}_x$. In these notations, the metaharmonic potential of the body T_t at the point z_t of unit density is written in the form

$$V_t^\varepsilon(z_t) = \int_{T_t^\varepsilon} \frac{e^{-\varkappa|r_t - \varepsilon n_x|}}{|r_t - \varepsilon n_x|}\, dy.$$

Theorem 1. *The solution $\zeta(x)$ of the stated problem, under the hypotheses of the main theorem, satisfies the nonlinear integro-differential equation*

$$2\pi\zeta(x) - \int_S \frac{\partial}{\partial n_x}\left(\frac{e^{-\varkappa r_{xy}}}{r_{xy}}\right)\zeta(y)\, d_y\sigma = f + g\zeta + \Phi(\zeta) + \Psi(\zeta), \tag{1}$$

where

$$f = -\frac{\partial}{\partial \nu}(V_1 - V)\Big|_{\nu=0}, \qquad g = -\frac{\partial^2}{\partial \nu^2}(V_1 - V)\Big|_{\nu=0},$$

$$\Phi(\zeta) = \frac{\partial V_1}{\partial \nu}\Big|_{\nu=0} + \zeta\frac{\partial^2 V}{\partial \nu^2}\Big|_{\nu=0} - \frac{\partial V_1}{\partial \nu}\Big|_{\nu=\zeta},$$

$$\Psi(\zeta) = \sum_{n=2}^{\infty} Z_n,$$

where

$$Z_n = \lim_{\varepsilon\to 0} \frac{1}{n!} \left\{ \left[\frac{\partial^n}{\partial t^n} \left(\frac{\partial V_t^\xi}{\partial \nu} \right) \right]_{t=0} \right\}, \qquad n = 2, 3, \dots,$$

is an integro-power form with respect to ξ, ξ_ξ, ξ_η.

We shall now prove the converse: every sufficiently small solution of equation (1) is a solution of the problem.

Let ξ be a function prescribed on the surface S and satisfying equation (1). Lay off the values ξ along the outward normal at each point x of the surface S, and denote by $\hat{S}_1$ the surface thus obtained; denote by $\hat{T}_1$ the body bounded by this surface, and by $\hat{V}_1$ its metaharmonic potential of unit density.

Theorem 2. If ξ is a sufficiently small solution of equation (1), then the equality

$$\hat{V}_1 \equiv V_1$$

holds, i.e. the body T_1, bounded by the surface $\nu = \xi(\xi, \eta)$, has V_1 as its exterior potential.

4°. The solution of equation (1) and the investigation of the properties of the operator $\Psi(\xi)$ have been carried out by us on the basis of the method of V. K. Ivanov (1).

To obtain the properties of the operator $\Psi(\xi)$, estimates are derived and used for the boundary values of metaharmonic potentials and their derivatives. In doing so, the theory of differentiable mappings developed by L. Lichtenstein (3,4), by means of introducing a complex parameter, is used.

For the operator $\Psi(\xi)$ the following is valid.

Theorem 3. If the function $\xi \in R_2$, defined on the surface S, has a sufficiently small norm, then $\Psi(\xi)$ belongs to the space R_2, and for any $\xi, \hat{\xi}$ from R_2 satisfying the conditions

$$\|\xi\| \le \omega, \qquad \|\hat{\xi}\| \le \omega,$$

the inequalities

$$\|\Psi(\xi)\| \le a_1 \omega^2,$$

$$\|\Psi(\xi) - \Psi(\hat{\xi})\| \le b_1 \omega \|\xi - \hat{\xi}\|$$

hold, where a_1, b_1, d are positive constants, $0 < \omega \le d$.

Write equation (1) in the form

$$A\xi = f + g\xi + P(\xi), \tag{2}$$

where

$$A\xi = 2\pi\xi(x) - \int_S \frac{\partial}{\partial n_x}\left(\frac{e^{-\varkappa r_{xy}}}{r_{xy}}\right)\xi(y)\,d_y\sigma, \qquad P(\xi) = \Phi(\xi) + \Psi(\xi).$$

If $\xi \in R_2$ and the norm of ξ is sufficiently small, then all terms in (2) belong to R_2, and the operator A has in R_2 a continuous inverse A^{-1}. Consequently, (2) can be replaced by the equivalent equation

$$\xi = A^{-1}f + A^{-1}g\xi + A^{-1}P(\xi). \tag{3}$$

Let ω be a positive number satisfying the inequalities

$$\omega \le d, \qquad \omega(1 + b\|A^{-1}\|) < 1, \qquad \omega^2 - 2(1 + 2a\|A^{-1}\|)\omega + 1 > 0,$$

where a, b, d are positive constants depending on the form of the domain T.

Theorem 4. If the norms of the functions f and g satisfy the estimates

$$\|f\| \le \frac{\omega}{\|A^{-1}\|}, \qquad \|g\| \le \frac{\omega}{\|A^{-1}\|},$$

then equation (3) has a unique solution satisfying the condition $\|\xi\| < d$, and this solution can be found by the method of successive-

successive approximations

$$\xi_1 = A^{-1}f,$$

$$\xi_{n+1} = A^{-1}f + A^{-1}g\xi_n + A^{-1}P(\xi_n), \qquad n = 1, 2, ...$$

Theorems 3 and 4, together with Theorems 1 and 2, lead to the conclusion that was formulated in the form of the main theorem.

In conclusion, I express my sincere gratitude to A. V. Bitsadze, under whose supervision this work was carried out.

Institute of Mathematics with Computing Center
of the Siberian Branch of the Academy of Sciences of the USSR

Received
13 VIII 1963

References

1. V. K. Ivanov, *Izv. AN SSSR, Ser. Mat.*, **20**, 793 (1956).

2. L. N. Sretenskii, *Izv. AN SSSR, Ser. Mat.*, **2**, 551 (1938).

3. L. Lichtenstein, *Gleichgewichtsfiguren der rotierenden Flüssigkeiten*, Berlin, 1933.

4. L. Lichtenstein, *Vorlesungen über einige Klassen nichtlinearer Integralgleichungen*, Berlin, 1931.

Note: Figure translations are in progress. See original paper for figures.