CYBERNETICS AND CONTROL THEORY

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Fig. 1

Figure 1: Fig. 1

Abstract

Full Text

CYBERNETICS AND CONTROL THEORY

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A CRITERION FOR THE ABSOLUTE STABILITY OF PULSE AUTOMATIC SYSTEMS WITH MONOTONE CHARACTERISTICS OF THE NONLINEAR ELEMENT

(Presented by Academician V. S. Kulebakin, 23 X 1963)

The criterion for the absolute stability of nonlinear pulse automatic systems (NPAS), obtained in ([1,2]), makes it possible to establish sufficient conditions for the absolute stability of NPAS with a nonlinear characteristic belonging to the sector $(0, k)$. In a number of cases this criterion gives small values of the upper bound of the sector k. To increase this bound, in ([3,4]) it was proposed to introduce a lower bound of the sector $r > 0$. In the present paper another possibility is considered, consisting in the introduction of an additional restriction on the derivative of the nonlinear characteristic. In contrast to ([5,6]), these restrictions are less stringent from the practical point of view, and the formulation of the criterion is very simple.

Fig. 1

Consider an NPAS consisting of a nonlinear element (NE) and a linear pulse part (LPP) ([1]).

Assume that the linear part is stable, and that the characteristic of the NE belongs to the sector $(0, k)$ and is monotone, i.e., satisfies the conditions (Fig. 1):

$$\text{a) } \Phi(0) = 0; \qquad \text{b) } 0 < \Phi(x)/x < k; \qquad \text{c) } \Phi'(x) \geq 0. \tag{1}$$

The equations of the NPAS may be represented in the form

$$x[n] = f[n] - \sum_{m=0}^{n} w[n-m]\Phi(x[m]), \tag{2}$$

where $w[n]$ is the impulse response of the LPP, and $x[n]$ is the error.

Introduce the auxiliary functions ([1])

$$\varphi_N[n] = \begin{cases} \Phi(x[n]), & 0 \le n \le N, \\ 0, & n < 0,\ n > N; \end{cases} \tag{3}$$

$$\psi_N[n] = x_N[n] - \frac{1}{k}\varphi_N[n] + a\Delta x_N[n-1], \tag{4}$$

where

$$\Delta x_N[n-1] = x_N[n] - x_N[n-1], \tag{5}$$

$$x_N[n] = f\,[n] - \sum_{m=0}^{n} w[n-m]\varphi_N[m]. \tag{6}$$

The second auxiliary function $\psi_N[n]$ coincides with that considered in ([1]) for $a = 0$. Obviously, for $0 \le n \le N$

$$x_N[n] \equiv x[n], \tag{7}$$

where $x_N[n]$ is determined from (6).

Form the expression

$$\rho_N = \sum_{n=0}^{\infty} \varphi_N[n]\psi_N[n] = \sum_{n=0}^{N} \Phi(x[n])\psi_N[n], \tag{8}$$

which, by virtue of the Lyapunov–Parseval theorem, can also be rep-

write in the form

$$\rho_N = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_N^*(-j\bar{\omega})\Psi_N^*(j\bar{\omega})\,d\bar{\omega}, \tag{9}$$

where (7)

$$\begin{aligned} \Phi_N^*(j\bar{\omega}) &= D\{\varphi_N[n]\}_{q=j\bar{\omega}} = \sum_{n=0}^{\infty} e^{-j\omega n}\varphi_N[n], \\ \Psi_N^*(j\bar{\omega}) &= D\{\psi_N[n]\}_{q=j\bar{\omega}} = \sum_{n=0}^{\infty} e^{-j\omega n}\psi_N[n] \end{aligned} \tag{10}$$

and $\bar{\omega} = \omega T$.

Computing the spectral functions by formulas (10), taking (3)–(4) into account, and substituting them into (9), after transformations we obtain:

$$\rho_N = -\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\sqrt{\operatorname{Re}\Pi^*(j\bar{\omega})}\,\Phi_N^*(j\bar{\omega}) - \frac{F^*(j\bar{\omega})}{\sqrt{\operatorname{Re}\Pi^*(j\bar{\omega})}}\right|^2 + \frac{1}{8\pi}\int_{-\pi}^{\pi}\frac{|F^*(j\bar{\omega})|^2}{\operatorname{Re}\Pi^*(j\bar{\omega})}\,d\bar{\omega}, \tag{11}$$

where

$$\operatorname{Re}\Pi^*(j\bar{\omega}) = \left[1 + \alpha(1 - e^{-j\bar{\omega}})\right] W^*(j\bar{\omega}) + \frac{1}{k} > 0, \tag{12}$$

$$F^*(j\bar{\omega}) = \left[1 + \alpha(1 - e^{-j\bar{\omega}})\right] F_{\text{H}}^*(j\bar{\omega}). \tag{13}$$

In formulas (12), (13), $W^*(j\bar{\omega})$ is the frequency response of the linear pulse element corresponding to $w[n]$, and $F_{\text{H}}^*(j\bar{\omega})$ is the spectral function corresponding to $f_{\text{H}}[n]$.

Discarding in (11) the first negative integral, we obtain the inequality

$$\rho_N \leqslant \frac{1}{8\pi}\int_{-\pi}^{\pi}\frac{|F^*(j\bar{\omega})|^2}{\operatorname{Re}\Pi^*(j\bar{\omega})}\,d\bar{\omega} = C(\alpha). \tag{14}$$

By virtue of (12), $C(\alpha) > 0$ and does not depend on N.

Now replacing in the expression for ρ_N (8) $\psi_N[n]$ by its value from (4) and taking (7) into account, we represent inequality (14) in the form

$$\rho_N = \sum_{n=0}^{N}\Phi(x[n])\,x[n]\left(1 - \frac{\Phi(x[n])}{kx[n]}\right) + \alpha\sum_{n=0}^{N}\Phi(x[n])\,\Delta x[n-1] < C(\alpha). \tag{15}$$

Let us estimate the second sum in (15). Represent $\Phi(x)$ by Taylor' s formula in the form

$$\Phi(x) = \Phi(x[n]) - \Phi'(\xi_n)(x[n] - x), \tag{16}$$

where ξ_n is some "mean" value.

Integrating (16) within the limits from $x[n-1]$ to $x[n]$, we obtain

$$\int_{x[n-1]}^{x[n]} \Phi(x)\, dx = \Phi(x[n])\, \Delta x[n-1] - \Phi'(\xi_n) \frac{(\Delta x[n-1])^2}{2}. \tag{17}$$

By virtue of condition (1), the second term in (17) is negative. Discarding it, we obtain the inequality:

$$\int_{x[n-1]}^{x[n]} \Phi(x)\, dx \leqslant \Phi(x[n])\, \Delta x[n-1]. \tag{18}$$

Summing both sides of (18) over n from 1 to N, we obtain

$$\sum_{n=1}^{N} \int_{x[n-1]}^{x[n]} \Phi(x)\, dx = \int_{x[0]}^{x[N]} \Phi(x)\, dx \leq \sum_{n=1}^{N} \Phi(x[n])\, \Delta x[n-1],$$

whence

$$\sum_{n=0}^{N} \Phi(x[n])\, \Delta x[n-1] \geq \Phi(x[0])x[0] + \int_{x[0]}^{x[N]} \Phi(x)\, dx. \tag{19}$$

Replacing in (15) the second sum by the smaller value from (19), we strengthen inequality (15), which, after adding to both sides

$$\alpha \int_{0}^{x[0]} \Phi(x)\, dx,$$

takes the form:

$$\sum_{n=0}^{N} \Phi(x[n])x[n] \left(1 - \frac{\Phi(x[n])}{kx[n]}\right) + \alpha\Phi(x[0])x[0] + \alpha \int_{0}^{x[N]} \Phi(x)\, dx \leq C(\alpha) + \alpha \int_{0}^{x[0]} \Phi(x)\, dx = C_1(\alpha). \tag{20}$$

For $\alpha \geq 0$, all terms on the right- and left-hand sides of the inequality are positive. Thus, $C_1(\alpha) > 0$ and does not depend on N. Therefore, as N increases, the first sum of positive (by virtue of (16)) terms will be bounded. According to the well-known theorem on convergence of series with positive terms, we conclude that

$$\lim_{n\to\infty} \Phi(x[n])x[n] \left(1 - \frac{\Phi(x[n])}{kx[n]}\right) = 0, \tag{21}$$

whence, taking into account (16), we obtain

$$\lim_{n\to\infty} x[n] = 0, \tag{22}$$

which indicates absolute stability. Thus, we arrive at the following theorem:

Theorem 1. An NIAS possessing a monotone nonlinear characteristic belonging to the sector $(0, k)$, and a stable LCh, will be absolutely stable if there exists a number $\alpha \geq 0$ such that, for all $\bar{\omega}$ in the interval $0 \leq \bar{\omega} \leq \pi$, the inequality

$$\operatorname{Re} \Pi^*(j\bar{\omega}) = \operatorname{Re}\left[1 + \alpha(1 - e^{-j\bar{\omega}})\right] W^*(j\bar{\omega}) + \frac{1}{k} > 0 \tag{23}$$

is satisfied.

This theorem is a complete analogue of V. M. Popov' s theorem ([8]). But, unlike continuous systems, in an NIAS there is here an additional restriction on the derivative of the nonlinear characteristic: it must be monotone. We note that for practical cases this restriction is not substantial. Taking into account that $\bar{\omega} = \omega T$, as $T \to 0$, from (23) we obtain the known condition of absolute stability for continuous systems due to V. Popov ()

$$\operatorname{Re}(1 + \bar{\alpha} j\omega) W(j\bar{\omega}) + \frac{1}{k} > 0. \tag{24}$$

For a geometric interpretation of condition (23), let us represent it in the form

$$U^*(\bar{\omega}) - \alpha \tilde{V}^*(\bar{\omega}) + \frac{1}{k} > 0, \tag{25}$$

where

$$U^*(\bar{\omega}) = \operatorname{Re} W^*(j\bar{\omega}), \qquad \tilde{V}^*(\bar{\omega}) = \operatorname{Re}\left[e^{-j\bar{\omega}} W^*(j\bar{\omega})\right] - \operatorname{Re} W^*(j\bar{\omega}). \tag{26}$$

The curve

$$W^*(j\bar{\omega}) = U^*(\bar{\omega}) + jV^*(j\bar{\omega}) \tag{27}$$

corresponds to the frequency characteristic of the LPC. The curve

$$\widetilde{W}^*(j\bar{\omega}) = U^*(\bar{\omega}) + j\widetilde{V}(\bar{\omega}) \tag{28}$$

will be called, as in (8), the modified frequency characteristic. It can easily be constructed from $W^*(j\bar{\omega})$ on the basis of relations (25). Let us draw, in the plane of the modified frequency characteristic $(U^*, \widetilde{V}^*)$, the straight line

Fig. 2

Figure 2: Fig. 2

Fig. 3

Figure 3: Fig. 3

$$U^{*}-\alpha\widetilde{V}^{*}+\frac{1}{k}=0, \tag{29}$$

passing through the point $-1/k$ of the real axis with slope $1/\alpha$ (Fig. 2). Then condition (25), and hence also the conditions of Theorem 1, are satisfied if the modified frequency characteristic lies entirely in that part of the plane which contains the origin, i.e. to the right of the straight line (29). To obtain the greatest value of k, the straight line (29) must be tangent to the modified frequency characteristic (Fig. 2).

Fig. 2

Fig. 3

Using the considerations given in ([3],[4]), one can generalize to the case of a neutral and unstable LPC and obtain the following theorem:

Theorem 2. An NPAS possessing a nonlinear characteristic monotone with respect to a ray and belonging to the sector (r,k) (Fig. 3) will be absolutely stable if the LPC enclosed by feedback with gain coefficient r is stable and if there exists a number $\alpha\geq 0$ such that, for all $\bar{\omega}$ in the interval $0\leq\bar{\omega}\leq\pi$, the inequality

$$\operatorname{Re}\frac{\left[1+\alpha(1-e^{-j\bar{\omega}})\right]W^{*}(j\bar{\omega})}{1+rW^{*}(j\bar{\omega})}+\frac{1}{k-r}>0 \tag{30}$$

is satisfied.

In the particular case $\alpha=0$ we arrive at the previously established conditions for absolute stability of an NPAS ([1–3]). For $\alpha>0$, the upper limiting value of the sector (r,k) can be increased in comparison with the case $\alpha=0$.

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Note: Figure translations are in progress. See original paper for figures.