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Abstract

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ON A METHOD OF BASIS EXPANSIONS

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Let us consider, in a bounded domain Ω $(P \in G,\ 0 \le t \le t_0)$, $[P = P(x_1, \dots, x_m)]$, of the $(m+1)$-dimensional space $(x_1, \dots, x_m, t)$, the equation

$$\Lambda_P[u] = u_{tt} + \frac{a}{t} u_t + c(t) u \quad (a = 2\beta = \text{const}), \tag{1}$$

where $\Lambda_P[u]$ is a linear differential operator with respect to the function $u(P,t)$, independent of t. In applications (1–3), (1) also occurs in the form $\Lambda_P[v] = v_{tt} + B(t) v_t$, where $vT = t^\beta u$, $TB(t) = 2T'$, $c(t) = \beta(\beta-1)t^{-2} - T''/T$, and $T(t)$, $B(t)$, and $c(t)$ are defined by series in integral or fractional powers of t. We shall call the integrals of the equation

$$\Lambda_P[z] = z_{tt} + \frac{a}{t} z_t, \tag{2}$$

for which the relation

$$t z_t(P,t,a) = (a-1)[z(P,t,a-2) - z(P,t,a)] \tag{3}$$

holds, basis functions $z(P,t,a)$.

We shall seek $u(P,t)$ in the form of a series in the basis functions $g_n(P,t) \equiv z(P,t,a+2n)$:

$$u(P,t) = \sum_{n=0}^{\infty} A_n(t) g_n(P,t). \tag{4}$$

Then, with the aid of the equality $t(g_n)_t = (a+2n-1)(g_{n-1} - g_n)$, we obtain

$$A_0'(t) = 0; \qquad 2(a+2n+1)\left(A_{n+1}' - \frac{n+1}{t} A_{n+1}\right) =$$

$$= 2(a+2n-1)\left(A_n' - \frac{n}{t} A_n\right) - tA_n'' - aA_n' - tc(t)A_n \quad (n = 0, 1, 2, \dots). \tag{5}$$

Integrating (5) with respect to t, we arrive at the recurrence sequence

$$A_0(t) = 1; \qquad 2(a+2n+1)A_{n+1} = (a+3n-2)A_n - tA_n' +$$

$$+(n-1)(2-a-n)t^{n+1}\int t^{-n-2}A_n(t)\,dt - t^{n+1}\int t^{-n}c(t)A_n(t)\,dt. \qquad (6)$$

The values $A_n(t)$ determined by these equalities do not depend on the number of dimensions m or on the form of the operator $\Lambda_P[u]$. Moreover, if $c(t) = 0$, then, putting $n = 1, 2, 3, ...$ in (6), we successively find $A_1 = 0$, $A_2 = 0, ...$, so that the series (4), for $c(t) = 0$, indeed terminates at its first term $z(P, t, a)$. Formulas (6) make it possible once and for all to tabulate the values $A_1, A_2, ...$ for those $c(t)$ which occur in applications. Thus, for example, in the well-known Chaplygin equation [(3)], where $a = 1/3$,

$$c(t) = \sum_{n=-1}^{\infty} c_n t^{2n/3}, \qquad \eta = -\left({}^3/_2\, t\right)^{2/3},$$

restricting ourselves to the principal term $c(t) \cong c_{-1}t^{-2/3}$, we find $A_1 = -\frac{1}{2}c_{-1}({}^3/_2)^{2/3}\eta^2$, $A_2 = \frac{1}{20}c_{-1}^2({}^3/_2)^{4/3}\eta^4$, $A_3 = \frac{1}{4}A_2 - \frac{3}{640}c_{-1}^3\eta^6$. In another important case, when $c(t) = b^2 = \text{const}$, we obtain

$$n!(\beta + {}^1/_2)_n A_n(t) = (-1)^n (bt/2)^{2n} \quad (n = 0, 1, 2, ...). \qquad (7)$$

We give several examples illustrating the application of the series (4) to the solution of singular problems for equations of hyperbolic and parabolic types.

1. Suppose that the operator $\Lambda_P[u]$ admits (for $a \geq 0$) existence and uniqueness of the solution of the singular Cauchy problem:

$$u(P, 0) = f(P), \qquad u_t(P, 0) = 0, \qquad f(P) \in C^2(G). \qquad (8)$$

Choose as a basis $\{z(P, t, a)\}$ the solution of the same problem (8) for equation (2), and assume that in a neighborhood of the line $t = 0$, $c(t) = O(t^\mu)$ $(\mu > -1)$. Then, for $a > a_1 \geq 0$, $\mu_0\Gamma(\beta-\beta_1)\Gamma(\beta_1+1/2) = 2\Gamma(\beta+1/2)$, and any $m = 1, 2, ...$, the series (4) generates the connection formula:

$$u(P, t, a) = \mu_0 \int_0^1 z(P, \xi t, a_1)\xi^{a_1}(1-\xi^2)^{\beta-\beta_1-1}R_0(\xi, t)\,d\xi, \qquad (9a)$$

$$R_0(\xi, t) = \sum_{n=0}^{\infty} [(\beta + 1/2)_n/(\beta - \beta_1)_n] A_n(t)(1-\xi^2)^n. \qquad (9b)$$

Conversely, the transformation

$$u(P,t,a) = z(P,t,a) + \int_0^1 z(P,\xi t,a)\xi^a R_1(\xi,t)\,d\xi \tag{10a}$$

leaves the parameter a unchanged, where

$$R_1(\xi,t) = \sum_{n=1}^{\infty} 2\big[(\beta+1/2)_n/(n-1)!\big] A_n(t)(1-\xi^2)^{n-1}. \tag{10b}$$

Since, by virtue of (6), for $\mu > -1$ the $A_n(t)$ have the initial data

$$A_{n+1}(0) = 0, \qquad A_n'(0) = 0 \qquad (n = 0, 1, 2, ...), \tag{11}$$

we have $R_0(\xi,0) = 1$, $R_{0t}(\xi,0) = R_1(\xi,0) = R_{1t}(\xi,0) = 0$, which ensures fulfillment of the conditions (8) for the functions (9a) and (10a). In particular, when $c(t) = b^2$, we arrive at the connections found in ([4], [5]); here (7), (9b), and (10b) give $R_0 = \bar{J}_{\beta-\beta_1-1}(\zeta)$, $R_1\sqrt{1-\xi^2} = -btJ_1(\zeta)$, $\zeta = bt\sqrt{1-\xi^2}$.

Let $f(P) \in C^2(G)$, and let Λ_P be the Laplace operator $\Lambda_P[u] = \Delta_P[u] = \sum_{i=1}^m u_{x_i x_i}$. Then, for $a_1 = m-1$, $a > m-1 > 0$, (9a) is transformed into the solution of the problem (1), (8):

$$u(P,t) = \mu_1 \int_0^1 M(P,\xi t; f)\,\xi^{m-1}(1-\xi^2)^{(a-m-1)/2} R_2(\xi,t)\,d\xi. \tag{12a}$$

Here $M(P,t;f) = z(P,t,m-1)$ is the mean value of the function $f(P)$ over the sphere of radius t with center at the point P, $\mu_1\Gamma(m/2)\Gamma(\beta+1/2-m/2) = 2\Gamma(\beta+1/2)$, and

$$R_2(\xi,t) = \sum_{n=0}^{\infty} \big[(\beta+1/2)_n/(\beta+1/2-m/2)_n\big] A_n(t)(1-\xi^2)^n. \tag{12b}$$

An analogous representation also holds in the plane case ($m = 1$, $\Lambda_P[u] = u_{xx}$), where, putting $\gamma_1\Gamma^2(\beta) = \Gamma(a)$, $\beta > 0$, we find

$$u(x,t) = \gamma_1 \int_0^1 f[x + t(2\xi-1)][\xi(1-\xi)]^{\beta-1} R_3(\xi,t)\,d\xi, \tag{13a}$$

$$R_3(\xi,t) = \sum_{n=0}^{\infty} \big[(\beta+1/2)_n/(\beta)_n\big] A_n(t)[4\xi(1-\xi)]^n. \tag{13b}$$

For $c(t) = b^2$, the series (12b), (13b) have infinitely large domains of convergence and, by virtue of (7), reduce to the holomorphic functions $R_2 = \bar{J}_{a-m-1}(\zeta)$,

$$R_3 = \bar{J}_{\beta-1}\left[2bt\sqrt{\xi(1-\xi)}\right],$$

indicated in ([5],[6]). In the general case, when $c(t) = O(t^\mu)$ $(\mu > -1,\ t \to 0)$, (12b) and (13b) converge to functions continuous and bounded at all points $(\xi, t) \in D$ $(0 \le \xi \le 1,\ 0 \le t \le t_0)$.

2. Let now $m = 1,\ \Lambda_P[u] = u_{xx},\ 0 \le t \le x$. We use as a basis $\{z(x, t, a)\}$ the solutions of the singular Tricomi problem:

$$z(x, 0) = f(x) \in C^2(0 \le x \le x_0), \qquad z(x, x) = 0, \qquad f(0) = 0, \tag{14}$$

which are also connected by the recurrence relation (3). Then, starting from (4) and putting $a > 0,\ \mu_2 = 2^{2-a}\gamma_1 \cos \beta\pi$, we arrive at the solution of problem (1), (14):

$$u(x, t) = \mu_2 t^{1-a} \int_0^{x-t} f(\xi)\left[(x-\xi)^2 - t^2\right]^{\beta-1} R_4(\xi, x, t)\, d\xi, \tag{15a}$$

$$R_4(\xi, x, t) = \sum_{n=0}^{\infty} (-1)^n \left[(\beta + \tfrac{1}{2})_n / (\beta)_n\right] t^{-2n} A_n(t) \left[(x-\xi)^2 - t^2\right]^n, \tag{15b}$$

where now, if $c(t) = b^2$, then (7), (15b) give

$$R_4 = \bar{I}_{\beta-1}\left[b\sqrt{(x-\xi)^2 - t^2}\right] \ ([6]).$$

3. Finally, let us turn to the parabolic case $m = 1,\ \Lambda_P[u] = u_x$, and consider in the domain $x \ge 0,\ t \ge 0$ the mixed singular problem

$$u(0, t) = 0\ (t \ge 0), \qquad u(x, 0) = f(x)\ (x \ge 0), \qquad f(0) = 0. \tag{16}$$

Its solution for arbitrary noninteger a is the integral

$$u(x, t) = \frac{1}{\Gamma(\frac{1}{2} - \beta)} \left(\frac{t}{2}\right)^{1-a} \int_0^x f(\xi)(x-\xi)^{\beta-3/2} e^{-t^2/4(x-\xi)} Q_1(\xi, x, t)\, d\xi,$$

$$Q_1(\xi, x, t) = \sum_{n=0}^{\infty} (-1)^n (\beta + \tfrac{1}{2})_n A_n(t) \left[4(x-\xi)/t^2\right]^n. \tag{17}$$

This time, for $a > a_1 \geq 0,\ \mu_3\Gamma(\frac{1}{2}-\beta)\Gamma(\beta-\beta_1) = 2\Gamma(\frac{1}{2}-\beta_1)$, the relations

$$u(x,t,a) = \mu_3 \int_1^\infty z(x,\xi t,a_1)\xi^{a_1}(\xi^2-1)^{\beta-\beta_1-1}Q_2(\xi,t)\,d\xi, \tag{18a}$$

$$Q_2(\xi,t) = \sum_{n=0}^{\infty}(-1)^n\left[(\beta+\tfrac{1}{2})_n/(\beta-\beta_1)_n\right]A_n(t)(\xi^2-1)^n. \tag{18b}$$

If, however, the parameter a remains unchanged, then

$$u(x,t,a) = z(x,t,a) + \int_1^\infty z(x,\xi t,a)\xi^a Q_3(\xi,t)\,d\xi, \tag{19a}$$

$$Q_3(\xi,t) = \sum_{n=0}^{\infty} 2(-1)^{n+1}\left[(\beta+\tfrac{1}{2})_{n+1}/n!\right]A_{n+1}(t)(\xi^2-1)^n. \tag{19b}$$

In particular, when $c(t) = -b^2$, from (7), (17), and (18b) we find $Q_1 = \exp[b^2(\xi-x)]$, $Q_2 = \bar{J}_{\beta-\beta_1-1}\left(bt\sqrt{\xi^2-1}\right)$.

Having determined from the Volterra-type integral equations (9), (10), (18) and (19) $z(P,t,a_1)$, we arrive at inverse relations which, in the particular case $a_1 = m-1,\ \Lambda_P = \Delta_P$, give the inversion of formula (12a) with respect to the means $M(P,t;f)$. If $z(P,t,a)$ is the solution of the Cauchy problem (2), (8) for $a > m-1,\ \Lambda_P = \Delta_P$, and $f(P) \in C^2(G)$ is a subharmonic func-

positive in the domain G ($\Delta_P f \geqslant 0$ or $f''(x) \geqslant 0$, when $m \geqslant 2$ or $m = 1$), then, as shown in (7), $z_t(P,t) \geqslant 0$ everywhere in Ω, and $z(P,t,a)$ tends uniformly to $f(P)$ as $a \to \infty$. Thus, in the series (4) $\lim_{n\to\infty} g_n = f(P)$, and, moreover, from (3) and the condition $z_t \geqslant 0$ it follows that $g_n \geqslant g_{n+1}$ $(n = 0,1,2,...)$, so that (4) is an expansion in terms of a nonincreasing sequence of basis functions $g_n(P,t)$, bounded below.

We arrive at the same conclusions also in the mixed problems (14), (16), but here, for their validity, it is sufficient to require that $f'(x) \geqslant 0$ $(x \geqslant 0)$. With the aid of the expressions found, one can (more simply than in ([3], [8–10])) compute corrections to the known Poisson–Appell solutions, and also estimate the error in the approximate replacement of $u(P,t)$ by $z(P,t)$.

Of special interest in problems of transonic gas dynamics are the automodel solutions of the two-dimensional Euler–Poisson equation. They arise if one sets in (13) $R_3 = 1$, $f(x) = x^\nu$, which gives

$$g_n^{(\nu)}(x,t) = x^\nu\, {}_2F_1(-\nu/2,\ 1/2-\nu/2,\ \beta+n+1/2;\ t^2/x^2).$$

Such integrals $g_0^{(\nu)}(x,t)$ were used earlier for $\nu = n$, $\nu = 1/3$, $\nu = -5/3$, and other values of ν ([1], [3], [11]). Passing from $c(t) = 0$ to the Chaplygin equation, one can, by formulas (13), compute correction terms $A_n g_n^{(\nu)}$ for all these functions $g_0^{(\nu)}(x,t)$. In an analogous way, in the parabolic case

$$z_x = z_{tt} + \frac{a}{t} z_t$$

with the aid of (17), corrections are determined for the automodel solutions considered earlier in ([2], [9]) for $a = 1/3$ and $a = 0$. Along with (1) and (2), the method of basis expansions (4) is also applicable to more general pairs of equations of the same or of two different types with singular coefficients. Thus, for example, instead of (2) the equation

$$\Delta_P[z] = z_{tt} + \frac{a}{t} z_t + b^2 z,$$

which also has solutions with the recurrent dependence (3), may serve as the basis.

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REFERENCES

1. S. V. Fal' kovich, *Prikl. matem. i mekh.*, **25**, No. 2, 218 (1961).

2. V. G. Levich, *Fiziko-khimicheskaya gidrodinamika*, M.–L., 1959.

3. P. Germain, G. Giraud, R. Bader, ONERA Publ., No. 60 (1952); P. Germain, G. Giraud, ibid., No. 102, 47 (1961).

4. I. N. Vekua, *Novye metody resheniya ellipticheskikh uravnenii*, M.–L., 1948.

5. M. N. Olevskii, DAN, **101**, No. 1, 21 (1955).

6. M. B. Kapilevich, DAN, **81**, No. 1, 13 (1951); **91**, No. 4, 719 (1953).

7. A. Weinstein, *Ann. Mat. pura ed appl.*, **43**, 325 (1957).

8. S. Bergman, R. Bojanić, *Arch. Rat. Mech. and Anal.*, **40**, No. 4, 323 (1962).

9. J. Hadamard, *C. R.*, **152**, No. 18, 1148 (1911).

10. J. Diaz, G. Ludford, *Ann. mat. pura ed appl.*, **38**, 33 (1955).

11. F. I. Frankl, *Inzh. zhurn.*, **1**, No. 1, 29 (1961).

Note: Figure translations are in progress. See original paper for figures.