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Abstract

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G. I. KALMYKOV

ON THE REVERSIBILITY OF FELLER PROCESSES

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Let $\{\xi_t\}$ be a one-dimensional stationary Markov process on the open interval (r_1, r_2), which may be infinite on one or both sides. We shall assume that there exists a transition-probability density of the process $f(t, x, y)$.

By C we denote the space of functions f continuous on the interval (r_1, r_2) and such that $f(x)$ tends to finite limits as $x \to r_j$, where $j = 1, 2$. In the space C we define the norm

$$\|f\| = \sup_{r_1 < x < r_2} |f(x)|.$$

$\mathcal{L}(r_1, r_2)$ is the space of all integrable functions g with norm

$$\|g\| = \int_{r_1}^{r_2} |g(x)|\, dy.$$

A Markov process is called a Feller process if the space C is invariant with respect to the transformation

$$T_t f(x) = \int_{r_1}^{r_2} f(y) f(t, x, y)\, dy, \quad t > 0.$$

In what follows we shall consider stationary Feller processes. The density of the stationary distribution will be denoted by $p(x)$.

Let $\{\xi_t\}$ be a one-dimensional stationary Markov process possessing the density of the joint distribution

$$p(t, x, y) = p(x) f(t, x, y)$$

of the random variables ξ_t and ξ_{s+t}. The process under consideration is called reversible if the relation

$$p(t, x, y) = p(t, y, x)$$

holds.

The purpose of this article is to investigate under what conditions a stationary Feller process is reversible. In doing so we shall rely on A. M. Yaglom' s theorem on the reversibility of stationary Markov processes ([3]), §5, theorem 2). This theorem was proved by him for time-homogeneous processes satisfying the forward and backward diffusion equations, whose transition-probability density is uniquely determined by these equations. As applied to a one-dimensional stationary Markov process, this theorem may be formulated as follows:

Theorem. A reversible one-dimensional Markov process satisfying the diffusion equations

$$\frac{\partial u(t,x)}{\partial t} = a(x)\frac{\partial^2 u(t,x)}{\partial x^2} + b(x)\frac{\partial u(t,x)}{\partial x}, \tag{1}$$

$$\frac{\partial v(t,y)}{\partial t} = \frac{\partial^2}{\partial y^2}[a(y)v(t,y)] - \frac{\partial}{\partial y}[b(y)v(t,y)], \tag{2}$$

has a stationary distribution with density $p(y)$, everywhere different from zero, satisfying the equation

$$[a(y)p(y)]'_y - b(y)p(y) = 0. \tag{3}$$

Conversely, every one-dimensional Markov process subject to the diffusion equations (1) and (2) and having a stationary distribution with density satisfying equation (3) will be reversible.

We shall always assume that in equations (1) and (2) the function $a(x)$ is positive, and $a'(x)$ and $b(x)$ are continuous in the interval (r_1, r_2).

Introduce the function

$$W(x) = \exp\left(-\int_{x_0}^{x} b(s)a^{-1}(s)\,ds\right),$$

where $x_0 \in (r_1, r_2)$ is fixed.

The boundary r_j is called **regular** if $W(x) \in \mathcal{L}(x_0, r_j)$, $a^{-1}(x)W^{-1}(x) \in \mathcal{L}(x_0, r_j)$; it is called **exit** if

$$a^{-1}(x)W^{-1}(x) \notin \mathcal{L}(x_0, r_j), \qquad W(x)\int_{x_0}^{x} a^{-1}(s)W^{-1}(s)\,ds \in \mathcal{L}(x_0, r_j).$$

In all other cases the boundary is called **unattainable** ([5,6]). In particular, a regular boundary r_j is called **reflecting** if all solutions of equation (1) satisfy the boundary condition

$$\lim_{x\to r_j}\left[W^{-1}(x)\frac{\partial}{\partial x}u(t,x)\right] = 0. \tag{4}$$

In W. Feller' s paper ([2], §13) it is proved that the diffusion process in the interval (r_1, r_2) governed by equation (1) is Feller. It should be noted that the transition probability density of a Feller stationary process governed by equations (1) and (2) is always determined uniquely. From W. Feller' s results it follows that nonuniqueness could arise only in the case where one or both boundaries are regular ([1], §23). However, since the process is stationary, the solution of equation (3) must preserve the norm. This is possible only in the following cases: a) one boundary r_j is regular and the other is unattainable, and the solutions of equation (2) satisfy the boundary condition

$$\lim_{y\to r_j}\left\{\frac{\partial}{\partial y}[a(y)v(t,y)] - b(y)v(t,y)\right\} = 0; \tag{5}$$

b) both boundaries are regular, and the solutions of equation (3) satisfy, for $j = 1, 2$, the boundary conditions (5). But these boundary conditions uniquely determine the transition probability density.

By Ω we shall denote the operator

$$a(x)\frac{\partial^2}{\partial x^2} + b(x)\frac{\partial}{\partial x}.$$

The domain of definition of the operator Ω will be denoted by $D(\Omega)$. Obviously, $D(\Omega)$ is dense in the space C.

Theorem 1. *If each of the boundaries is either unattainable or reflecting, then the stationary Feller process governed by the backward diffusion equation (1) is also governed by the forward diffusion equation (2) and is reversible.*

Proof. We shall prove that, in the cases considered by us, the process is governed by the forward equation (2), and the density of its stationary distribution satisfies equation (3). Then, by A. M. Yaglom' s theorem on the reversibility of Markov processes, it will be reversible.

Consider the **first case**: both boundaries are unattainable. As proved in W. Feller' s paper ([1], §7), without loss of generality one may assume that $a(x) \equiv 1$. If $a(x) \not\equiv 1$, then, by a corresponding change of scale, one can make $a(x)$ identically equal to one. Then equations (1), (2), and (3) respectively take the form:

$$\frac{\partial u(t,x)}{\partial x} = \frac{\partial^2 u(t,x)}{\partial x^2} + b(x)\frac{\partial u(t,x)}{\partial x}; \tag{1'}$$

$$\frac{\partial v(t,y)}{\partial t} = \frac{\partial^2 v(t,y)}{\partial y^2} - \frac{\partial}{\partial y}[b(y)v(t,y)]; \tag{2'}$$

$$p'(y) - b(y)p(y) = 0. \tag{3'}$$

Further we shall assume that $a(x) \equiv 1$.

From the results of W. Feller (see (1), § 15) it follows that in this case there exists a unique regular (†) Green function $K(x, s)$ for the operator Ω.

The integral operator R_λ with kernel $K(x, s)$, defined in the space C, is the unique operator inverse to the operator $(\lambda E - \Omega)$. Consequently, the unique infinitesimal operator for which the operator Ω is its extension will be the operator Ω itself.

The adjoint infinitesimal operator is the restriction of the operator

$$\Omega^* = \frac{d}{dx}\left\{\frac{d}{dx} - b(x)\right\}$$

to the set of functions satisfying the condition

$$\lim_{x \to r_j} (g' - bg) = 0, \qquad j = 1,\, 2. \tag{6}$$

In other words, the process under consideration is governed by equation $(2')$. The density of the stationary distribution $p(y)$ belongs to the domain of definition of the adjoint infinitesimal operator. Consequently, $p(y)$ satisfies the condition

$$\lim_{y \to r_j} [p'(y) - b(y)p(y)] = 0, \qquad j = 1,\, 2, \tag{7}$$

and also the equation

$$p''(y) - [b(y)p(y)]' = 0. \tag{8}$$

Integrating (8) once and taking into account the conditions (7), we obtain

$$p'(y) - b(y)p(y) = 0, \qquad r_1 < y < r_2. \tag{3'}$$

Thus, if the boundaries are inaccessible, then a stationary Feller process governed by equation $(1')$ is also governed by equation $(2')$, and the density of its stationary distribution satisfies equation $(3')$. By the reversibility theorem of A. M. Yaglom, the process is reversible.

Consider the second case: both boundaries are reflecting. In this case ((2), theorem 10), the forward diffusion equation is equation (2); solutions of equation (2) are characterized by the additional conditions

$$\lim_{y \to r_j} \left\{\frac{\partial}{\partial y}[a(y)v(t, y)] - b(y)v(t, y)\right\} = 0, \qquad j = 1,\, 2. \tag{9}$$

Since $p(y)$ is a solution of equation (2) with the initial condition $v(0, y) = p(y)$, $p(y)$ satisfies the equation

$$[a(y)p(y)]'' - [b(y)p(y)]' = 0. \tag{10}$$

Substituting the function $p(y)$ for $v(t, y)$ in (9), we obtain:

$$\lim_{y \to r_j} \{[a(y)p(y)]' - b(y)p(y)\} = 0, \qquad j = 1, 2. \tag{11}$$

Integrating (10) once and using (11), we obtain that $p(y)$ satisfies (3).

Since the process is governed by equations (1) and (2), and the density of its stationary distribution satisfies equation (3), it is reversible.

The third case. One of the boundaries is inaccessible and the other is reflecting. In this case the forward diffusion equation is also equation (2); solutions of equation (2) are characterized by the additional conditions (9). The proof is analogous to that given in the second case.

Theorem 2. *If a stationary Feller process is governed by the diffusion equations (1) and (2), then it is reversible and is the unique stationary Feller process governed by equations (1) and (2), and each of its boundaries is either inaccessible or reflecting.*

Proof. The density of the stationary distribution is a solution of equation (2). This solution preserves the norm. As was shown by W. Feller ((1), §§ 15, 23), in this case each of the boundaries is either inaccessible or reflecting. By Theorem 1 the process is reversible.

Since the process is reversible, the density of its stationary distribution satisfies equation (3). However, equation (3) has a unique nontrivial solution. Further, the transition-probability density of a stationary Feller process is uniquely determined by equations (1) and (2). Therefore there exists a unique stationary Feller process subject to equations (1) and (2).

O. V. Sarmanov (4) investigated stationary reversible Markov processes subject to equations (1) and (2). He found solutions of (1) and (2) for cases in which $a(x)$ and $b(x)$ are given by polynomials ((4), § 3). From the results of W. Feller ((2), § 13) it follows that these processes are Feller processes. This can also be proved directly from the definition of a Feller process. By Theorem 2, in all the cases considered by O. V. Sarmanov ((4), § 3), the stationary Feller process is uniquely determined by its equations (1) and (2).

Mathematical Institute named after V. A. Steklov
Academy of Sciences of the USSR

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Note: Figure translations are in progress. See original paper for figures.