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Abstract

Full Text

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ON HÖLDER CONTINUITY OF SOLUTIONS
AND THEIR DERIVATIVES FOR LINEAR
AND QUASILINEAR EQUATIONS OF ELLIP-
TIC AND PARABOLIC TYPES

(Presented by Academician V. I. Smirnov, 2 I 1964)

One of the central parts of our works (^{1a-d}) is the study of Hölder continuity of solutions (and their derivatives) of linear equations of elliptic and parabolic types

$$\mathcal{L}_1 u \equiv \frac{\partial}{\partial x_i} (a_{ij}(x)u_{x_j} + a_i(x)u) + b_i(x)u_{x_i} + c(x)u = f(x), \quad (1)$$

$$\mathcal{L}_2 u \equiv \frac{\partial u}{\partial t} - \frac{\partial}{\partial x_i} (a_{ij}(x, t)u_{x_j} + a_i(x, t)u) + b_i(x, t)u_{x_i} + c(x, t)u = f(x, t), \quad (2)$$

quasilinear equations with divergence principal part

$$\mathcal{L}_3 u \equiv \frac{\partial}{\partial x_i} (a_i(x, u, u_x)) + a(x, u, u_x) = 0, \quad (3)$$

$$\mathcal{L}_4 u \equiv \frac{\partial u}{\partial t} - \frac{\partial}{\partial x_i} (a_i(x, t, u, u_x)) + a(x, t, u, u_x) = 0, \quad (4)$$

certain classes of systems of such equations and general quasilinear equations

$$\mathcal{L}_5 u \equiv a_{ij}(x, u, u_x)u_{x_i x_j} + a(x, u, u_x) = 0, \quad (5)$$

$$\mathcal{L}_6 u \equiv u_t - a_{ij}(x, t, u, u_x)u_{x_i x_j} + a(x, t, u, u_x) = 0 \quad (6)$$

and the derivation of estimates of the Hölder norms $|\cdot|_\alpha$ for u and u_x in terms of constants characterizing the functions entering the equations. With the aid of specially constructed examples we showed that the conditions under which this is done in (^{1a-d}) are, in a certain sense, unimprovable.

Our approach to obtaining estimates of Hölder norms (under the assumption that the solution under consideration is bounded and possesses a certain smoothness) consists, in general outline, of the following. We prove that: a) any solution of equations (1) or (3), as well as any of its derivatives with respect to x_k , is a function of the so-called class $\mathfrak{B}_m(\Omega, \dots)$; b) any solution of equations (2) or (4) and any of its derivatives with respect to x_k is a function of the class $\mathfrak{B}_2(Q_T, \dots)$; c) the gradient with respect to x of any solution of equation (5) is a vector-function belonging to the class $\mathfrak{B}_2^{N_1}(\Omega, \dots)$, and d) the gradient with respect to x of any solution of equation (6) is a vector-function belonging to the class $\mathfrak{B}_2^{N_1}(Q_T, \dots)$. The proof of all these facts, once they have been discovered, is simple and is based on a special choice of an arbitrary function entering the integral identities that replace equations (1)–(6), and on the application of the Hölder and Young inequalities. A function belongs to one of the classes \mathfrak{B} if it satisfies certain inequalities containing free parameters.

For these functions—elements of \mathfrak{B} —we prove that they are Hölder continuous and that their norms can be estimated in terms of the numerical parameters entering the definition of the classes \mathfrak{B} . As applied to differential equations, this fact gives, in view of what was said above, the possibility

estimate the Hölder norms of u and u_x . Thus, Hölder continuity and the existence of these estimates are due to a more general fact than being a solution (or its derivative) of equations of elliptic or parabolic type, namely—belonging to the classes \mathfrak{B} .

However, the proof of Hölder continuity of functions of the classes \mathfrak{B} is not simple, especially in the case when t enters. In view of this, it was desirable to find a simpler way of obtaining estimates of Hölder norms. We shall give it here, using several propositions from our papers that are not difficult to prove, and, for the nonstationary case, also a lemma (see below). We also make use of Moser’s idea ⁽²⁾ of considering, together with the solution of an equation, its logarithm, which is a “subsolution.” The final results are the same as those previously obtained by us in ⁽¹⁻⁾. Therefore we shall not formulate them all, but shall illustrate the proposed method on the example of the equation

$$u_t - \frac{\partial}{\partial x_i} (a_{ij}(x, t)u_{x_j}) = 0 \quad (7)$$

and indicate those propositions from ⁽¹⁾, Part III, on which it is based for the general case of equations (1)–(6). We shall restrict ourselves to an interior estimate; for estimates near the boundary one must take the corresponding lemmas from ⁽¹⁾, Part III.

Thus, let $u(x, t)$ be a solution of equation (7), or, equivalently, of the corresponding integral identity

$$\int (u_t \eta + a_{ij} u_{x_j} \eta_{x_i}) dx = 0, \quad (8)$$

where η is a smooth function, finite in the domain under consideration, and let

$$\nu \sum_{i=1}^n \xi_i^2 \leq a_{ij}(x, t) \xi_i \xi_j \leq \mu \sum_{i=1}^n \xi_i^2, \quad \nu, \mu = \text{const} > 0. \quad (9)$$

As is known (see ⁽¹⁾), in order to prove the Hölder continuity of $u(x, t)$ and to obtain an estimate of its Hölder norm $|u|_\alpha$, it is sufficient to show that the oscillation of u in an arbitrary cylinder of standard form is strictly greater than its oscillation in a smaller coaxial cylinder with the same vertex. From dimensional considerations for equations (7), it is sufficient to prove that for any solution $u(x, t)$ of it, defined in the cylinder $Q_2 = K_2 \times [0, a] = \{|x| \leq 2, 0 \leq t \leq a\}$ and varying in the interval $[0, 1]$, and for the cylinder $Q_1 = K_1 \times [3/4 a, a] = \{|x| \leq 1, 3/4 a \leq t \leq a\}$, the inequality

$$\text{osc}\{u, Q_1\} \leq \eta \text{osc}\{u, Q_2\} = \eta, \quad (10)$$

holds, where a and η are certain constants determined only by n, ν , and μ from (9), with $\eta < 1$. We shall prove (10), using for this only Lemma 7 from ⁽¹⁾, Part I, Lemma 2 from ⁽¹⁾, Part III, and the following lemma:

Lemma 1. For any function $w(x)$ from $W_2^1(K_2)$, for which

$$\text{mes}\{x \in K_1, \text{ where } w(x) \leq 0\} \geq b \text{mes } K_1, \quad b > 0,$$

and for any smooth function $\mathfrak{N}(x) = \mathfrak{N}(|x|)$, decreasing monotonically from 1 to 0 as $|x|$ varies from 0 to 2, and equal to 1 for $0 \leq |x| \leq 3/2$, the inequality

$$\int_{\{x \in K_2, w > 0\}} w^2(x) \mathfrak{N}^2(x) dx \leq \beta \int_{\{x \in K_2, w > 0\}} w_x^2(x) \mathfrak{N}^2(x) dx \quad (11)$$

holds with a constant β depending only on n and b .

Consider $u(x, 0)$ in the ball K_1 . At least either

$$\text{mes}\{x \in K_1, u(x, 0) \geq 1/2\} \geq 1/2 \text{mes } K_1, \quad (12)$$

or $\text{mes}\{x \in K_1, u(x, 0) \leq 1/2\} \geq 1/2 \text{mes } K_1$. If the first case occurs

then in all further arguments we shall work with the function $u(x, t)$. In the opposite case we consider $1 - u(x, t)$.

Thus, let (12) be true. Substitute in (8) $\eta(x, t) = \xi^2(x) \max\{u(x, t) - k, 0\}$, where $\xi(x)$ is a smooth function finite in the sphere $K_\rho \subset K_1$ of radius ρ , with

$0 \leq \xi(x) \leq 1$, and k is an arbitrary number. After simple estimates (see (1 $^\Gamma$), part I) we obtain

$$\frac{1}{2} \frac{\partial}{\partial t} \int_{A_{k,\rho}(t)} [u(x,t) - k]^2 \xi^2(x) dx + \nu \int_{A_{k,\rho}(t)} u_x^2 \xi^2 dx \leq \frac{4\mu^2}{\nu} \int_{A_{k,\rho}(t)} (u-k)^2 \xi_x^2 dx, \quad (13)$$

where $A_{k,\rho}(t)$ is the set of points x of K_ρ at which $u(x,t) > k$.

By Lemma 7 of (1 $^\Gamma$), part I, from (13) and (12) there follows* the existence of such positive constants b and a (here and below all constants depend only on ν, n , and μ) that

$$\text{mes}\{x \in K_1, u(x,t) \geq 1/8\} \geq b \text{mes} K_1, \quad t \in [0, a]. \quad (14)$$

Consider in Q_2 the function $v(x,t) = -\psi(u(x,t))$, where $\psi(u) = -\ln 8u$. It is nonnegative where $u(x,t) \geq 8^{-1}$. If we prove that in Q_1 $v(x,t)$ does not exceed some constant M , then in Q_1 we shall have $8u \geq e^{-M}$, and hence also (10).

Substituting in (8) $\eta = \psi'(u)\zeta(x,t)$, where ζ is a smooth function finite in K_2 , we obtain the identity

$$\int_{K_2}' (v_t \zeta + a_{ij} v_{x_i} v_{x_j} \zeta + a_{ij} v_{x_j} \zeta_{x_i}) dx = 0, \quad (15)$$

similar to (8). From it follows inequality (13) for v . By virtue of Lemma 2 of (1 $^\Gamma$), part III, from (15) there follows the desired upper estimate for v in Q_1 , if an upper estimate is known for $\|v(x,t)\|_{L_2(K_{5/4})}$ for $t \in [a/2, a]$.

For this purpose put in (15) $\zeta = \mathfrak{N}^2(x)\chi^2(t)$, where $\mathfrak{N}(x)$ is a function of the form indicated in Lemma 1, and χ is equal to 0 for $0 \leq t \leq 8^{-1}a$, to $8ta^{-1} - 1$ for $8^{-1}a \leq t \leq 4^{-1}a$, and to 1 for $4^{-1}a \leq t \leq a$, and integrate with respect to t from 0 to a . Hence, and from Lemma 1, as a result of simple estimates we obtain

$$\int_{K_{3/4}} \int_{a/4}^a (v^2 + v_x^2) dx dt \leq c(\nu, \mu, n). \quad (16)$$

Now put in (15) $\zeta = v(x,t)\xi^2(x)$, where $\xi(x)$ is a function finite in $K_{3/4}$, and integrate the result first with respect to t over the interval $[t', t]$, and then with respect to t' over the interval $a/4 \leq t' \leq a/2$, assuming $t \geq a/2$. As a result of estimates by Cauchy's inequality and using (16), this gives the desired estimate for $\|v\|_{L_2(K_{5/4})}$ for $t \in [a/2, a]$.

Thus (10) is established. For the case of equations of general form (1)–(6), the scheme of proof is the same, and all the stages for carrying it out are contained in the papers (1^{a-Γ}). For the elliptic case it is still simpler, since for it the estimate $\|v\|_{L_2}$ is obtained at once from an identity of type (15) and the known estimate $\int u_x^2 dx$ for the solution. The boundedness of v from above follows from Lemma 2 of (1^Γ), part III. For the elliptic case, the last lemma admits a certain strengthening and is proved more simply (see (1^Γ), part III), namely:

* Since in (13) there is no second term on the right of the form $\gamma \text{mes}^{1-2/q} A_{k,\rho}$, and k is arbitrary, the requirements on k and H indicated in Lemma 7 must be discarded.

Lemma 2. If $u(x) \in W_m^1(\Omega)$, $m > 1$, and u for any $K_\rho \subset \Omega$ and any number $k \geq k_0$ satisfies the inequality

$$\int_{A_{k,\rho}} |u_x|^m \xi^m dx \leq \gamma \left[\int_{A_{k,\rho}} (u-k)^m |\xi_x|^m dx + k^m (\text{mes } A_{k,\rho})^{1-m/q} \right], \quad q > n,$$

where $\xi(x)$ is a function finite in K_ρ and $0 \leq \xi(x) \leq 1$, then for $\Omega' \subset \Omega$

$$\max_{\Omega'} v(x)$$

is estimated from above by a constant depending only on $k_0, n, m, \gamma, q; \|u\|_{L_m(\Omega)}$, and the distance from Ω' to the boundary of Ω .

Let us formulate one of the results on solutions of equations (3).

Theorem. Let $u(x)$ be a generalized solution from $W_m^1(\Omega)$, $m > 1$, of equation (3), with $\|u\|_{L_\infty(\Omega)} = \text{vrai max}_\Omega |u| \leq M$. Suppose that for $x \in \Omega$, $|u| \leq M$, the functions $a_i(x, u, p)$, $a(x, u, p)$ are measurable and satisfy the inequalities

$$a_i(x, u, p)p_i \geq \nu |p|^m - \varphi_0(x), \quad \nu = \text{const} > 0,$$

$$|a_i(x, u, p)| \leq \mu [\varphi_1(x)|p|^{m-1-\delta_1} + \varphi_3(x)], \quad \mu = \text{const},$$

$$|a(x, u, p)| \leq \mu [\varphi_2(x)|p|^{m-\delta_2} + \varphi_4(x)],$$

where

$$\varphi_0, \varphi_4 \in L_{q/m}(\Omega), \quad \varphi_3 \in L_{q/(m-1)}(\Omega), \quad \varphi_1 \in L_{q/\delta_1}(\Omega), \quad \varphi_2 \in L_{q/\delta_2}(\Omega), \quad q > n,$$

$$0 \leq \delta_1 \leq m-1, \quad 0 \leq \delta_2 \leq m.$$

Then $u \in C_{0,\alpha}(\Omega)$, and for any $\Omega' \subset \Omega$ the norm $|u|_{\alpha,\Omega'}$ is estimated from above by a constant depending only on $M, n, \nu, \mu, \|\varphi_0, \varphi_4\|_{L_{q/m}(\Omega)}, \|\varphi_3\|_{L_{q/(m-1)}(\Omega)}, \|\varphi_1\|_{L_{q/\delta_1}(\Omega)}, \|\varphi_2\|_{L_{q/\delta_2}(\Omega)}$, and the distance from Ω' to the boundary of Ω . If S satisfies condition A (see (1')), $u|_S \in C_{0,\beta}(S)$, then $u \in C_{0,\alpha}(\bar{\Omega})$, $0 < \alpha \leq \beta$, and $|u|_{\alpha,\Omega}$ is estimated in terms of the constants indicated above (they also determine $\alpha > 0$), $|u|_{\beta,S}$, and the constants from condition A .

From this theorem, in particular, follows the Hölder continuity of solutions of equations (1) under admissible singularities in the coefficients (for (1), $m = 2$). In (1) it is clarified under what conditions the boundedness $\|u\|_{L_\infty(\Omega)}$ can be replaced by boundedness of $\|u\|_{L_q(\Omega)}$.

For equations (5) and (6), the Hölder continuity of u_{x_i} is established as follows: one considers their combinations

$$w_{\pm}^l = 5nM^{-1}(\pm u_{x_l} + M) + \sum_{i=1}^n 4M^{-2}(u_{x_i} + M)^2, \quad l = 1, \dots, n,$$

where $M = \max_{i, Q_T} |u_{x_i}|$. For each pair of standard domains Q_ρ and $Q_{2\rho}$ (balls or cylinders), one of these functions w_+^l or w_-^l is chosen, with index l , for which

$$\omega_l = \text{osc}\{u_{x_l}, Q_{2\rho}\} \geq \max_i \text{osc}\{u_{x_i}, Q_{2\rho}\},$$

and for it it is proved that either $\omega_l \leq C\rho$, or its oscillation in Q_ρ constitutes a proper part of ω_l . This is done according to the scheme described above for equation (7), using an identity of type (8), derived in (1), part III, for each of the functions w_{\pm}^l (see (8.2)) from (1), part III. From the same behavior of the oscillations of w_{\pm}^l one derives the Hölder continuity of u_{x_i} and an estimate of their norm $|u_{x_i}|_\alpha$ (this was done in (1), part III, pp. 172-174).

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1. O. A. Ladyzhenskaya, N. N. Ural' tseva, a) DAN, **135**, No. 6 (1960); b) UMN, **16**, issue 1 (1961); c) DAN, **139**, No. 3 (1961); d) Izv. AN SSSR, Ser. Mat., **26**, No. 5, part I (1962); **26**, No. 5, part II (1962); **27**, No. 1, part III (1963); e) Vestn. LGU, No. 1 (1963).
2. J. Moser, Comm. Pure and Appl. Math., **14**, No. 3 (1963).

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