SECOND SOUND IN A WEAKLY NONIDEAL BOSE GAS

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Abstract

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In the work of N. N. Bogolyubov ([1]), in the hydrodynamic approximation (the frequency of oscillations of the system is much less than the frequency of collisions between particles), expressions were obtained for one-particle Green' s functions having poles corresponding to two types of elementary excitations: ordinary sound and second sound. In the present work the same problem is considered in another limiting case, when the collision frequency is small.

The chain of equations for the one-particle Green' s function $\langle\langle a_q;\, a_q^+\rangle\rangle = -i\theta(t-t')\langle[a_q(t); a_q^+(t')]\rangle$ has the form

$$i\frac{d}{dt}\langle\langle a_q;\, a_q^+\rangle\rangle = \delta(t-t') + \left(\frac{q^2}{2m} - \mu + \frac{N}{V}v(0)\right)\langle\langle a_q;\, a_q^+\rangle\rangle +$$

$$+\frac{\sqrt{N_0}}{V}v(q)\langle\langle \rho_q;\, a_q^+\rangle\rangle + \frac{1}{V}\sum_{k\neq 0,q} v(k)\langle\langle \rho_k a_{q-k};\, a_q^+\rangle\rangle,$$

$$i\frac{d}{dt}\langle\langle \rho_q;\, a_q^+\rangle\rangle = \delta(t-t')\sqrt{N_0} + \langle\langle \rho'_q;\, a_q^+\rangle\rangle, \tag{1}$$

$$i\frac{d}{dt}\langle\langle \rho'_q;\, a_q^+\rangle\rangle = \delta(t-t')\frac{q^2}{2m}\sqrt{N_0} + E_q^2\langle\langle \rho_q;\, a_q^+\rangle\rangle + \langle\langle \rho''_q;\, a_q^+\rangle\rangle +$$

$$+\frac{1}{V}\sum_{k\neq 0,q} v(k)\frac{q\cdot k}{m}\langle\langle \rho_k\rho_{-k+q};\, a_q^+\rangle\rangle,$$

where μ is the chemical potential; $v(q)$ is the Fourier component of the interaction potential between particles $U(x)$;

$$\rho_q = \sum_p a_p^+ a_{p+q}; \qquad \rho'_q = \sum_p \left(\frac{q^2}{2m} + \frac{p\cdot q}{m}\right) a_p^+ a_{p+q}; \qquad \rho''_q = \sum_p \frac{p\cdot q}{m}\,\frac{p+q\cdot q}{m}\, a_p^+ a_{p+q};$$

$$E_q^2 = \left(\frac{q^2}{2m}\right)^2 + \frac{q^2}{m}\frac{N}{V}v(q);$$

N is the total number of particles in the volume V; N_0 is the number of particles in the condensate ($N_0 \sim N$ at temperature θ below the phase-transition temperature). In deriving the last of equations (1), the operator $\sum_p a_p^+ a_p$ was replaced by the average number of particles N.

We shall assume that the interaction is small and the gas density is large. In this case, in equations (1) the sums over states with $k \neq 0, q$ may be discarded.

The chain of equations obtained after discarding the integral terms (the chemical potential is found from the equation $\langle i\, da_0/dt\rangle = 0$ and is then equal to $\mu = \frac{N}{V}v(0)$) is completely equivalent to the random-phase approximation and can be obtained directly from equation (2) of work ([2]). If in this chain, in the last equation, one discards the term $\langle\langle \rho_q'', a_q^+\rangle\rangle$, proportional to the average kinetic energy per particle, regarding it as small in comparison with the corresponding potential energy, one immediately arrives at the expressions for the Green' s functions (5) and (6) of work ([2]).

We shall now consider the term $\langle\langle \rho_q'', a_q^+\rangle\rangle$ as a small perturbation. In calculating the Green' s functions, instead of the operators a_q, ρ_k, ρ_k' it is convenient to introduce new operators α_q and β_q (see relations (11) of work ([2])), writing them in the form

four-component vector A_q—a column with components $\alpha_q,\ \alpha_{-q}^+,\ \beta_q,\ \beta_{-q}^+$, where

$$\alpha_q = \left(1 - \frac{N_0}{N}\right)^{-1/2} \left(a_q - \frac{\sqrt{N_0}}{2N}\left(\rho_q + \frac{2m}{q^2}\rho_q'\right)\right), \qquad \beta_q = \frac{1}{2\sqrt{N}}\frac{1}{u_q + v_q}\left(\rho_q + \frac{1}{E_q}\rho_q'\right)$$

and $u_q^2 = 1 + v_q^2 = \frac{1}{2}[1 + (q^2/2m + Nv(q)/V)/E_q]$. In the adopted approximation the vector A_q satisfies the equation

$$i\frac{dA_q}{dt} = L_{qA}q - \Lambda_q \frac{m}{q^2}\frac{\rho_q''}{\sqrt{N}}, \tag{2}$$

where L_q is a diagonal matrix with elements $q^2/2m,\ -q^2/2m,\ E_q$, and $-E_q$; Λ_q is a column with components $(N_0/N)^{1/2}(1 - N_0/N)^{-1/2},\ -(N_0/N)^{1/2}(1 - N_0/N)^{-1/2},\ -(u_q + v_q),\ u_q + v_q$; Λ_q^T is the corresponding row.

Let us introduce the fourth-order matrix Green's function $G_q(t - t') = \langle\langle A_q; A_q^+\rangle\rangle$, where $A_q^+(t') = (\alpha_q^+(t'), \alpha_{-q}(t'), \beta_q^+(t'), \beta_{-q}(t'))$, which combines all the Green' s functions of the zeroth approximation. For the function $G_q(t - t')$ we have the equations:

$$i\frac{dG_q}{dt} = \delta(t - t')J + L_{qG}q - \Lambda_q \frac{m}{q^2}\frac{1}{\sqrt{N}}\langle\langle \rho_q''; A_q^+\rangle\rangle,$$

$$i\frac{d}{dt}\langle\langle\rho''_q;A^+_q\rangle\rangle \equiv -i\frac{d}{dt'}\langle\langle\rho''_q;A^+_q\rangle\rangle = \delta(t-t')\langle[\rho''_q,A^+_q]\rangle + \tag{3}$$

$$+\langle\langle\rho''_q;A^+_q\rangle\rangle L_q - \langle\langle\rho''_q;\rho''^{+}_q\rangle\rangle\frac{1}{\sqrt{N}}\frac{m}{q^2}\Lambda^T_q,$$

where J is a diagonal matrix with elements $1,-1,1,-1$. For the Fourier transforms of the Green' s functions (see, for example, ([3])) from (3) we obtain

$$(E-L_q)G_q(E) = J - \Lambda_q\frac{m}{q^2}\frac{1}{\sqrt{N}}\langle\langle\rho''_q|A^+_q\rangle\rangle_E,$$

$$\langle\langle\rho''_q|A^+_q\rangle\rangle_E(E-L_q) = -\sqrt{N}\frac{q^2}{m}\eta_q(E)\Lambda^T_q, \tag{4}$$

where

$$\eta_q(E) = \frac{3}{N}\sum_p\frac{1}{m}\left(\frac{p\cdot q}{q}\right)^2 n_p + \left(\frac{m}{q^2}\right)^2\frac{1}{N}\langle\langle\rho''_q|\rho''^{+}_q\rangle\rangle_E. \tag{5}$$

Eliminating the function $\langle\langle\rho''_q|A^+_q\rangle\rangle$ with the aid of the equations obtained, we shall have

$$\{E-L_q-\eta_q(E)P_q[J+\eta_q(E)(E-L_q)^{-1}P_q]^{-1}\}G_q(E) = J, \tag{6}$$

where the matrix P_q is formed by multiplying the column Λ_q and the row Λ^T_q ($P_q = \Lambda_q\Lambda^T_q$). Equation (6), within the random-phase approximation, represents an exact relation of the Green' s functions entering the matrix $G_q(E)$ with the function $\langle\langle\rho''_q|\rho''_q\rangle\rangle_E$. Using the smallness of the quantity $\eta_q(E)$, proportional to the mean kinetic energy of a particle, we simplify expression (6), retaining only the first term in the expansion of the inverse matrix $G^{-1}_q(E)$ in $\eta_q(E)$. We obtain

$$G^{-1}_q(E) \cong J(E-L_q) - \eta_q(E)JP_{qJ}. \tag{7}$$

Inverting the matrix (7), it is easy to find expressions for the Green' s functions $\langle\langle\alpha_q|\alpha^+_q\rangle\rangle$, $\langle\langle\alpha^+_{-q}|\alpha^+_q\rangle\rangle$, $\langle\langle\alpha_q|\beta^+_q\rangle\rangle$, etc. Then, expressing by means of formulas (10) of work ([2]) the operators a_q,ρ_q,ρ'_q through α_q and β_q, we obtain

$$\langle\langle a_q|a^+_q\rangle\rangle = \Delta^{-1}_q(E)\left\{\left(1-\frac{N_0}{N}\right)\left[\left(E+\frac{q^2}{2m}+\chi\eta_q\right)\left(E^2-E^2_q-\frac{q^2}{m}\eta_q\right)+\frac{q^2}{m}\chi\eta^2_q\right]\right.$$

$$+\frac{N_0}{N}\left[\left(E+\frac{q^2}{2m}+\frac{N}{V}v(q)+\eta_q\right)\left(E^2-\left(\frac{q^2}{2m}\right)^2-\frac{q^2}{m}\chi\eta_q\right)-2\eta_q\left(E+\frac{q^2}{2m}\right)^2+\frac{q^2}{m}\chi\eta_q^2\right]\Bigg\}, \tag{8}$$

where $\chi = (N_0/N)(1-N_0/N)^{-1}$, and $\Delta_q(E)$ is the determinant of the matrix (7), equal to

$$\Delta_q(E) = \left(E^2-\left(\frac{q^2}{2m}\right)^2-\frac{q^2}{m}\chi\eta_q(E)\right)\left(E^2-E_q^2-\frac{q^2}{m}\eta_q(E)\right)-\left(\frac{q^2}{m}\right)^2\chi\eta_q^2(E). \tag{9}$$

Other Green functions are calculated analogously. For $\eta = 0$, expression (8) coincides with the corresponding formula of work ([2]). For $\theta = \theta_{\text{cr}}$ ($N_0 = 0$), (8) goes over into the expression for the Green function of an ideal gas. Expression (8), as a function of E, is defined in the upper half-plane ($\operatorname{Im} E > 0$).

In order to find the poles of the retarded function $G_q(E)$, all of which have a negative imaginary part and determine the asymptotic behavior of the function $G_q(t)$ as $t \to \infty$ ($G_q(t) \sim \exp(i\omega t - \gamma t)$, γ being the damping), it is necessary to analytically continue the determinant (9) of the matrix (7) into the lower half-plane ($\operatorname{Im} E < 0$) and set it equal to zero.

For small q, equation (9) has roots of phonon type. Putting $\varepsilon^2 \equiv E^2 - (q^2/2m)^2 \simeq c^2q^2$, for the square of the sound velocity c^2 we obtain the equation

$$c^2\left[c^2-\frac{N}{V}\frac{\nu(0)}{m}-\frac{1}{m}\eta(c)\left(1-\frac{N_0}{N}\right)^{-1}\right]+\frac{N_0}{N}\left(1-\frac{N_0}{N}\right)^{-1}\frac{N}{V}\frac{\nu(0)}{m}\frac{1}{m}\eta(c)=0,$$

where $\eta(c) = \lim_{q\to 0}\eta_q(cq)$.

(10)

Assuming that η/m is smaller than $N\nu(0)/Vm$ (see ([2])), we shall have

$$c_1^2 \simeq \frac{N}{V}\frac{\nu(0)}{m}+\frac{1}{m}\eta(c_1), \qquad c_2^2 \simeq \frac{N_0}{N}\left(1-\frac{N_0}{N}\right)^{-1}\frac{1}{m}\eta(c_2). \tag{11}$$

Here c_1 is the velocity of ordinary sound, which varies weakly with changes in temperature. The quantity c_2 corresponds to second sound and goes to zero at the critical point ($\theta = \theta_{\text{cr}}$, $N_0 = 0$). Relations (11) are equations which determine, generally speaking, the complex quantities c_1 and c_2, whose imaginary parts correspond to the damping of elementary excitations.

Let us consider the quantity $\eta_q(E)$ in more detail. Using the Green function $\langle\langle a_p^+ a_{p+q} \mid a_{p'+q}^+ a_{p'} \rangle\rangle$, found from the equation in the random-phase approximation (2) of work ([2]), and the definition of the operator ρ_q'', we find the Green function $\langle\langle \rho_q'' \mid \rho_q''^{+} \rangle\rangle_E$ in the region $\operatorname{Im} E > 0$. Substituting it into (5), we obtain the "exact" expression for $\eta_q(E)$:

$$\eta_q(E) = \varepsilon^2 \frac{m}{q^2} \varphi_q(\varepsilon) \left(\varepsilon^2 - \frac{q^2}{m} \frac{N}{V} \nu(q) \right) \left[\varepsilon^2 - \frac{q^2}{m} \frac{N}{V} \nu(q) (1 + \varphi_q(\varepsilon)) \right]^{-1}, \quad (12)$$

where

$$\varphi_q(\varepsilon) = \frac{1}{N} \sum_p n_p \left\{ \frac{m}{q^2} \varepsilon^2 \frac{m}{2pq} \ln \frac{\varepsilon^2 - \left(\frac{pq}{m}\right)^2 + \frac{q^2}{m}\frac{pq}{m}}{\varepsilon^2 - \left(\frac{pq}{m}\right)^2 - \frac{q^2}{m}\frac{pq}{m}} - 1 \right\}. \quad (13)$$

In the last expression, integration has been performed over the angular variable ϑ $(p \cdot q = pq \cos\vartheta)$. Putting $\varepsilon^2 \simeq c^2 q^2$ and passing to the limit as $q \to 0$, we obtain

$$\varphi(c) = \lim_{q \to 0} \varphi_q(cq) = \frac{1}{N} \sum_{p \neq 0} n_p \left(\frac{p}{m}\right)^2 \left[c^2 - \left(\frac{p}{m}\right)^2 \right]^{-1}, \qquad \operatorname{Im} c > 0. \quad (14)$$

Let us estimate the quantity $\varphi(c)$ at a temperature θ considerably below θ_{cr}. In the case of ordinary sound (see (11))

$$\varphi(c_1) \sim N^{-1} \sum n_p \left(\frac{p}{m}\right)^2 \left[\frac{N}{V} \frac{\nu(0)}{m} - \left(\frac{p}{m}\right)^2 \right]^{-1} \sim N^{-1} \sum n_p \frac{p^2}{m} \Big/ \frac{N}{V} \nu(0)$$

is, in the adopted approximation (see ([2])), a small quantity even at $\theta = \theta_{\text{cr}}$.

In the case of second sound,

$$c_2^2 \sim \frac{N_0}{N} \left(1 - \frac{N_0}{N}\right)^{-1} \frac{1}{N} \sum n_p \left(\frac{p}{m}\right)^2$$

and, consequently,

$$\varphi(c_2) \sim \frac{1}{N} \sum n_p \left(\frac{p}{m}\right)^2 \Big/ c_2^2 \sim \left(1 - \frac{N_0}{N}\right) \ll 1$$

for $\theta \ll \theta_{\text{cr}}$. Thus, in the low-temperature region, using the smallness of the quantity $\varphi(c)$, from (12) we obtain

$$\eta(c) \simeq mc^2\varphi(c) = \frac{a^3}{2\pi^2}\int_0^\infty p^2\,dp\,n_p\,\frac{p^2}{m}\,\frac{c^2}{c^2-(p/m)^2}, \qquad \operatorname{Im} c > 0, \tag{15}$$

where $a^3 = V/N$, and n_p are the mean occupation numbers calculated in the zeroth approximation (see formula (7) of paper [2]). At a temperature equal to zero, the quantity η, and consequently also the second-sound velocity c_2, will have a small but finite value due to the smearing of the condensate by the interaction. In this case, however, one should take into account in equations (1) the omitted integral terms, in the same way as was done in paper [4].

We shall assume that the temperature θ is small compared with θ_{cr}, but sufficiently large that the small contribution to η due to the interaction may be neglected, and we shall take for n_p the ideal-gas expression

$$n_p = \left(e^{p^2/2m\theta} - 1\right)^{-1}.$$

Expression (15) then simplifies. Continuing it into the lower half-plane, we shall have

$$\eta(c) = \frac{a^3}{2\pi^2}\frac{(2m\theta)^{5/2}}{m}\int_0^\infty \frac{x^4\,dx}{e^{x^2}-1}\,\frac{c^2}{c^2-\frac{2\theta}{m}x^2} - i\,\frac{a^3}{2\pi}\,\frac{m^4c^5}{e^{mc^2/2\theta}-1}, \qquad \operatorname{Im} c < 0. \tag{16}$$

Formulas (15) and (16) define the function $\eta(c)$, analytic in the entire complex c-plane.

Substituting expression (16) into (11), we obtain equations for the quantities c_1 and c_2, which determine the asymptotic behavior of the Green function $G_q(t)$ as $t \to \infty$. Thus we have here a typical "plasma" problem [5].

The velocity of ordinary sound c_1 is of order

$$\sqrt{\frac{N}{V}\frac{\nu(0)}{m}}.$$

Using the smallness of the parameter

$$\theta \Big/ \frac{N}{V}\nu(0),$$

in $\eta(c_1)$ we may put

$$c_1 = \sqrt{\frac{N}{V}\frac{\nu(0)}{m}}.$$

In doing so, one must take into account the imaginary part of the first term of expression (16), obtained on approaching the real axis from the lower half-plane. As a result, for c_1 we shall have the expression

$$c_1 = \pm\sqrt{\frac{N}{V}\frac{\nu(0)}{m}}\left\{1+\frac{V\theta}{N\nu(0)}\frac{a^3(2m\theta)^{3/2}}{2\pi^2}\int_0^\infty \frac{x^4\,dx}{e^{x^2}-1}\right\}-i\,\frac{a^3m^3}{8\pi}\left(\frac{N}{V}\frac{\nu(a)}{m}\right)^2 e^{-\frac{N}{V}\frac{\nu(a)}{2\theta}}. \tag{17}$$

The equation for c_2, obtained from (11) and (16), has the form

$$\int_0^\infty \frac{x^2\,dx}{e^{x^2}-1} = \int_0^\infty \frac{x^4\,dx}{e^{x^2}-1}\,\frac{1}{z^2-x^2} - i\pi\,\frac{z^3}{e^{z^2}-1}, \qquad \operatorname{Im} z < 0, \tag{18}$$

where

$$c_2^2 = \frac{2\theta}{m}z^2.$$

It follows from equation (18) that the real and imaginary parts of c_2 are quantities of the same order:

$$c_2 = s_2 - i\gamma_2,$$

where γ_2 is the damping,

$$s_2 \sim \gamma_2 \sim \sqrt{2\theta/m}.$$

Therefore, in the spectral intensities of the Green functions, the peak corresponding to second sound will be strongly broadened.

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Note: Figure translations are in progress. See original paper for figures.