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# Mathematics

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1964

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**Abstract**

**Full Text**

Mathematics

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## SINGULAR INTEGRAL EQUATIONS WITH A CONTINUOUS AND PIECEWISE CONTINUOUS SYMBOL

*(Presented by Academician V. I. Smirnov on 26 V 1964)*

Here the general method expounded in the paper <sup>(5)</sup> is used. We shall use all the concepts of <sup>(5)</sup> without qualification.

1°. **Class  $Q_p^n$ .** Let the set of linear (bounded) operators invariant with respect to shifts,\* acting from  $L_p^n(E_m)$  into  $L_p^n(E_m)$  ( $1 < p < \infty$ ), be denoted by  $N_p^n$ . It is known <sup>(1)</sup> that if  $A \in N_p^n$ , then  $A \in N_{p'}^n$ , where  $1/p + 1/p' = 1$ , and, consequently,  $A \in N_2^n$ . Considering  $A$  as an operator from  $L_2^n$ , we can pass to the Fourier transform, as a result of which the operator  $A$  becomes the operator of multiplication by a matrix  $\hat{A}$ , composed of essentially bounded functions. Following S. G. Mikhlin <sup>(3,4)</sup>, we shall call the matrix  $\hat{A}$  the **symbol of the operator**  $A$ . Thus the symbol is defined for every operator  $A$  ( $\in N_p^n$ ). The following class of operators will be important for us.

Introduce the subspace of the space  $N_p^n$  obtained by closing the set of homogeneous\*\* operators with an infinitely differentiable on  $S$  ( $S$  is the unit sphere) symbol, and denote it by  $Q_p^n$ . Obviously, the symbol of an operator from  $Q_p^n$  is a continuous matrix, homogeneous of degree zero, of order  $n$ . It can be shown that operators from  $Q_p^n$  are operators of local type.

**Lemma 1.** *The following three conditions are equivalent:*

I. *The operator  $A$  ( $\in Q_p^n$ ) is a local Noether operator at every point  $x_0$  ( $\in E_m$ ) (including  $x_0 = \infty$ ).*

II. *The operator  $A$  ( $\in Q_p^n$ ) is invertible.*

III. *The determinant of the symbol of the operator  $A$  is nonzero everywhere on  $S$ .*

In proving this proposition we used the results of <sup>(1)</sup>, p. 49, and applied the theory of normed rings with involution <sup>(2)</sup>, p. 73.

2°. **Generalized singular integral.**

**Definition 1.** An operator  $A$  ( $L_p^n(E_m) \rightarrow L_p^n(E_m)$ ) of local type will be called a **generalized singular integral** if at every point  $x$  ( $\in E_m$ ) it is equivalent to

an operator  $A_x$  ( $\in Q_p^n$ ). We shall call  $A_x$  the **principal part** of the operator  $A$  at the point  $x$ .

It turns out that:

- 1) the principal part is determined uniquely;
- 2)  $A$  depends continuously on  $x$ , i.e.  $\|A_x - A_{x_0}\| \rightarrow 0$  when  $x \rightarrow x_0$ ;
- 3) if  $A_x$  is a family of operators from  $Q_p^n$ , continuously depending on  $x$  ( $\in E_m$ ), then there exists a generalized singular integral  $A$ , having at each point  $x$  principal part  $A_x$ ; moreover,  $A$  is determined up to a completely continuous summand.

Property 3) is proved by applying Theorem 6 of <sup>(5)</sup>. It makes it possible to give meaning to singular integrals of the form (1) <sup>(5)</sup> also in the case when the cha-

\* An operator  $A$  is invariant with respect to shifts if it commutes with the operators  $h*$  of shifting the argument:

$$A(h * f) = h * Af.$$

\*\* An operator  $A$  is called homogeneous if it commutes with the operator  $\alpha*$  of multiplying the argument by a positive number  $\alpha$ :

$$\alpha * Af = A(\alpha * f).$$

the characteristic  $\Omega$  does not satisfy the sufficient conditions found up to the present time (see <sup>(6,7)</sup>), which ensure the existence of the singular integral (1) <sup>(5)</sup> in the usual sense of the word. We did not consider the question of whether, in this case as well, the integral (1) can be understood as an integral in the sense of the principal value.

**Definition 2.** Following S. G. Mikhlin <sup>(3,4)</sup>, we shall call the square matrix  $\Phi_A(x, \xi)$  of order  $n$ :  $\Phi_A(x, \xi) = \hat{A}_x(\xi)$ , where  $x \in E_m$ ,  $A_x$  is the principal part of the operator  $A$  at the point  $x$ ,  $\xi (\in S)$ , the **symbol of the generalized singular integral  $A$** .

Using Theorem 6 <sup>(5)</sup>, it is easy to establish that: 1) the symbol depends continuously on  $x, \xi$ ; 2) every matrix  $\Phi(x, \xi)$  ( $x \in E_m, \xi \in S$ ), continuously dependent on  $(x, \xi)$ , is the symbol of a generalized singular integral acting from  $L_2$  to  $L_2$ ; 3) the operator is determined by its symbol up to a completely continuous summand.

**Theorem 1.** *In order that an operator  $A$  be a Noether operator, it is necessary and sufficient that the symbol  $\Phi_A(x, \xi)$ , for all  $x(\in E_m)$  and  $\xi(\in S)$ , be different from zero.*

The theorem follows from Theorems 1 and 4 <sup>(5)</sup> and Lemma 1. All results <sup>(8,9)</sup> concerning the index are carried over completely automatically to the case of generalized singular integrals.

**3°. Definition 3.** We shall call a **composite generalized singular integral** an operator  $B(L_p^n(E_m) \rightarrow L_p^n(E_m))$  of the form

$$B = \sum_i A_i P_{D_i}, \quad (1)$$

where  $A_i$  are generalized singular integrals;  $D_i$  is a finite set of closed regions, not intersecting one another in interior points; the boundary  $\Gamma_i$  of the region  $D_i$  consists of a finite number of nonintersecting closed bounded surfaces of Lyapunov type;  $\bigcup_i D_i = E_m$ .

Let us denote  $\Gamma = \bigcup_i \Gamma_i$ . We shall carry out the investigation only for the case  $p = 2$ .

We shall call the **symbol of the operator**  $B$  the matrix  $\Phi_B(x, \xi)$ , defined for all points  $x (\notin \Gamma)$ ,  $\xi \neq 0$  and coinciding there with  $\Phi_{A_i}(x, \xi)$  when  $x (\in D_i)$ . It can be proved that  $\Phi_B(x, \xi)$  is continuous in its domain of definition. At points  $x_0$  of the boundary  $\Gamma$ ,  $\Phi_B(x, \xi)$  is not defined, but there exist two limiting values: one—if one passes to the limit while remaining in one region  $D_i$ , the other—if in another region  $D_j$ . We shall denote these limiting values by  $\Phi_B^i(x, \xi)$ ,  $\Phi_B^j(x, \xi)$ , respectively.

We introduce terms in which the principal results will be formulated. For this purpose we carry out the following constructions.

Let  $x_0$  be a point of the boundary  $\Gamma$  belonging to the regions  $D_i$  and  $D_j$ . Draw at the point  $x_0$  unit normals to the boundary  $\Gamma$ :  $n_{x_0}^i$  directed into the region  $D_i$  and  $n_{x_0}^j$  into  $D_j$ ;  $n_{x_0}^i$  and  $n_{x_0}^j$  mark on the unit sphere  $S$  ( $|\xi| = 1$ ) two points  $e_{x_0}^i$  and  $e_{x_0}^j$ . If at the point  $x_0$  the matrices  $\Phi_B^i(x_0, \xi)$ ,  $\Phi_B^j(x_0, \xi)$  are nonsingular for all  $\xi (\neq 0)$ , introduce for consideration the functional matrix  $G_{x_0}^{ij}(\xi) = [\Phi_B^j(x_0, \xi)]^{-1} \Phi_B^i(x_0, \xi)$  and the numerical matrix  $H_{x_0}^{ij} = G_{x_0}^{ij}(e_{x_0}^i)[G_{x_0}^{ij}(e_{x_0}^j)]^{-1}$ . Introduce the hyperplane  $\Pi_{x_0}$  orthogonal to the line joining the points  $e_{x_0}^i$  and  $e_{x_0}^j$ , and consider one more functional matrix  $G_{x_0, \xi}^{ij}(t) = G_{x_0}^{ij}(n_{x_0}^i t + \xi)$  ( $-\infty < t < +\infty$ ) for  $\xi (|\xi| = 1, \xi \in \Pi_{x_0})$ . For the case when no eigenvalue of the matrix  $H_{x_0}^{ij}$  is a negative real number, for the matrix

\* This theorem contains Theorem 7<sup>(5)</sup>. In analyzing the latter, a comparison was made with the results of S. G. Mikhlin<sup>(3,4)</sup>.

of  $G_{x_0, \xi}^{ij}$  one can define the quantity

$$\text{Ind } G_{x_0, \xi}^{ij}(t) = \frac{1}{2\pi} \left[ \left\{ \arg \Delta G_{x_0, \xi}^{ij}(t) \right\} \Big|_{t=-\infty}^{+\infty} + \sum_{k=1}^n \arg \lambda_k \right], \quad (2)$$

where  $\lambda_k$  are the eigenvalues of the matrix  $H_{x_0}^{ij}$ ; the branch of  $\arg \lambda_k$  is chosen so that  $|\arg \lambda_k| < \pi$ . The quantity (2) is, obviously, an integer. Let us note: when  $t$  varies from  $-\infty$  to  $+\infty$ ,  $G_{x_0, \xi}^{ij}(t)$  runs through the same values as the matrix  $G_{x_0}^{ij}(\xi)$  when  $\xi$  varies along the large semicircle of the unit sphere  $S$

connecting  $e_{x_0}^i$  with  $e_{x_0}^j$ , lying in the section of  $S$  by the plane passing through the segment  $e_{x_0}^i e_{x_0}^j$  and the point  $\xi$ . In view of the fact that in the case  $m > 2$  all these semicircles are homotopic, while for  $m = 2$  there are two nonhomotopic semicircles, expression (2): a) in the case  $m > 2$  depends neither on  $\xi$  nor on the order in the pair  $ij$  and will be denoted by  $\text{Ind}(x_0)$ ; b) in the case  $m = 2$  may take two values, which we denote by  $\text{Ind}^\pm(x_0)$ . Moreover, it is easy to prove that the matrix  $G_{x_0, \xi}^{ij}$  belongs to the class  $A^n(2)$  (see <sup>(11)</sup>, p. 37), and for it one can define a system of partial indices  $\chi_1(x_0, \xi), \dots, \chi_n(x_0, \xi)$  (see <sup>(11)</sup>, theorem 7).<sup>\*</sup> In this case the formula

$$\text{Ind } G_{x_0, \xi}^{ij}(t) = \sum_{k=1}^n \chi_k(x_0, \xi)$$

holds.

Partial indices, in contrast to the index, may depend on  $\xi$ . Let us further note that  $\text{Ind}(x_0)$  is effectively computable, whereas the partial indices, generally speaking, are not. In the case  $n = 1$  there is no need to introduce partial indices, since the single partial index coincides with the index.

**Theorem 2.** *In order that the composite generalized singular integral  $B(L_2^2(E_m) \rightarrow L_n^2(E_m))$  be a Noether operator, it is necessary and sufficient that:*

- 1) *the symbol matrix  $\Phi_B(x, \xi)$  and its limiting values on  $\Gamma$  be nondegenerate for no  $\xi$  ( $|\xi| = 1$ ) and  $x \in E_m$ );*
- 2)  $\text{Ind}(x_0) = 0$  ( $m > 2$ ),  $\text{Ind}^\pm(x_0) = 0$  ( $m = 2$ );
- 3)  $\chi_k(x_0, \xi) = 0$  ( $k = 1, \dots, n$ ) for all  $x_0 \in \Gamma$ ,  $\xi$  ( $\xi \in \Pi_{x_0}$ ,  $|\xi| = 1$ ).

**Remark.** Condition 2) is a consequence of condition 3). However, we have formulated it separately, since it is an effectively necessary (and in the case  $n = 1$  also sufficient) condition.

**Theorem 3.** *Suppose that condition 1) of Theorem 2 is satisfied. Then, in the case when all partial indices  $\chi_k(x_0, \xi)$  ( $k = 1, \dots, n$ ),  $x_0 \in \Gamma$ , are nonnegative (nonpositive) and do not change as  $\xi$  changes, there exists a right (left) regularizer and the range of the operator  $B$  is closed.*

**Theorem 4.** *Suppose that condition 1) of Theorem 2 is satisfied, as well as the following condition: for each pair of points  $x_0 \in \Gamma$ ,  $\xi$  ( $\xi \in \Pi_{x_0}$ ,  $|\xi| = 1$ ), the matrix  $G_{x_0, \xi}^{ij}(t)$  is representable in the form  $G_{x_0, \xi}^{ij}(t) = G_{x_0, \xi}^1 G_{x_0, \xi}^2(t) G_{x_0, \xi}^3$ , where  $G_{x_0, \xi}^1, G_{x_0, \xi}^3$  do not depend on  $t$ ;  $G_{x_0, \xi}^2(t)$  satisfies the condition  $G_{x_0, \xi}^2(t) + [G_{x_0, \xi}^2(t)]^* \geq \nu > 0$ . Here  $\nu$  is a constant depending only on  $\xi, x_0$ , but not on  $t$ ; the sign  $\geq$  denotes comparison of Hermitian matrices;  $*$  denotes Hermitian conjugation. Then  $B$  is a Noether operator.*

It makes sense to use Theorem 4 only in the case  $n > 1$ . In the case  $n = 1$ , exhaustive information is provided by Theorem 2. In proving Theorems 2, 3, and 4 we used the method of the paper <sup>(5)</sup>. The local investigation was carried out separately at points  $x \notin \Gamma$  and  $x \in \Gamma$ . At a point  $x(\notin \Gamma)$   $B$  is equivalent to an operator of the class  $Q_2^n$ , investigated in <sup>2</sup><sup>0</sup>. At a point  $x(\in \Gamma)$  the operator  $B$  is quasi-equivalent to an operator of the type

$$C = A_1 P_{E_m^+} + A_2 P_{E_m^-},$$

where  $A_1, A_2 \in Q_2^n$ ;  $E_m^+, E_m^-$  are half-spaces. The transformation establishing the quasi-equivalence consists in a local straightening of the boundary. The investigation

\* A detailed exposition of the paper <sup>(11)</sup> is given in <sup>(12)</sup>.

of the operator  $C$  is carried out by reduction to the Riemann boundary-value problem, to which the author applies the results of his papers <sup>(10-12)</sup>.

4°. **Generalized singular integral** on a manifold (g.s.m.). It is clear that the method of <sup>(5)</sup> is equally well adapted both to manifolds and to Euclidean spaces. Let  $X$  be a twice continuously differentiable closed Riemannian manifold of dimension  $m$ , with measure induced by the metric tensor; let  $\mathfrak{M}$  be an atlas of local charts.

**Definition 4.** An operator of local type ( $L_p^n(X) \rightarrow L_p^n(X)$ ) will be called a **generalized singular integral** (g.s.m.) if for every  $\varphi \in \mathfrak{M}$  there is a quasiequivalence

$$A \underset{\varphi}{\sim} \xrightarrow{\varphi(x)} A_{x,\varphi},$$

where

$$A_{x,\varphi} \in Q_p^n(L_p^n(E_m) \rightarrow L_p^n(E_m)).$$

By the **symbol** of  $A$  we shall mean the matrix

$$\Phi_\varphi(x, \xi) = \widehat{A}_{x,\varphi}(\xi).$$

In this case, as we see, the symbol also depends on  $\varphi$ . It is now clear how to define a composite g.s.m. on  $X$ . It is also clear that all the results 1°, 2°, 3° carry over to g.s.m.'s completely automatically (see Theorem 5 <sup>(5)</sup>).

**Remark.** A singular equation on a manifold with boundary s.k. is a special case of an equation with a composite g.s.m.

Thus, we have obtained necessary and sufficient conditions for the Noether property of s.k. in the space  $L_2$ , while Theorem 3, also in the absence of the Noether property, gives some additional information about s.k. S.k. were also investigated in <sup>(13)</sup> under other assumptions\*.

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Received  
17 V 1964

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\* In <sup>(13)</sup> an equation with an infinitely differentiable symbol was investigated for a single function (in our case the symbol may be discontinuous and an equation for vector functions is investigated). But the set of classes in which a solution is sought is considerably richer in <sup>(13)</sup> than in our case (we have only  $L_2$ ). Note <sup>(13)</sup> became known to us when the present article had already been prepared for publication.

*Note: Figure translations are in progress. See original paper for figures.*

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