MATHEMATICS

L. R. VOLEVICH

1964

SovietRxiv

Abstract

Full Text

MATHEMATICS

L. R. VOLEVICH

ON HYPOELLIPTIC SYSTEMS WITH VARIABLE COEFFICIENTS

(Presented by Academician I. G. Petrovskii, 24 I 1964)

According to L. Schwartz, a differential equation (system) is called **hypoelliptic** in a domain Ω if every generalized solution is infinitely differentiable whenever the right-hand side of the equation has this property.

L. Hörmander ([1]) established a criterion for the hypoellipticity of equations (systems) with constant coefficients. Naturally, the problem arose of extending these results to equations with variable coefficients. For a single equation ([2,3]) a naturally broad class was found (formally hypoelliptic operators)*. As for systems with variable coefficients, hypoellipticity had been proved only for certain classes (elliptic, p-parabolic, quasi-elliptic ([7])), possessing a special definite principal part. Below we propose a new class of hypoelliptic systems which includes all the above-mentioned systems and equations with variable coefficients.

1. Notation. $x = (x^1, \dots, x^n)$ is a point in Euclidean space R^n, $\xi = (\xi_1, \dots, \xi_n)$ are variables dual to x with respect to the scalar product $x \cdot \xi = x^1\xi_1 + \dots + x^n\xi_n$, $D = (D_1, \dots, D_n)$, $D_k = -i\,\partial/\partial x^k$. If $\alpha = (\alpha_1, \dots, \alpha_n)$ is an integer multi-index, then $D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$, $|\alpha| = \alpha_1 + \dots + \alpha_n$. A polynomial $Q(\xi)$ may be written in the form $\sum a_\alpha \xi^\alpha$ (a differential operator is written similarly as $Q(x; D) = \sum a_\alpha(x) D^\alpha$). Put $Q^{(\alpha)}(\xi) = \partial^{|\alpha|} Q / \partial\xi_1^{\alpha_1} \cdots \partial\xi_n^{\alpha_n}$; $\widetilde{Q}(\xi) = \left[\sum_\alpha |Q^{(\alpha)}(\xi)|^2\right]^{1/2}$. By $\mathcal{D}, \mathcal{D}(\Omega)$ we shall denote the space of infinitely differentiable functions in R^n, Ω, with compact supports, endowed with the natural topology. By $\mathcal{D}', \mathcal{D}'(\Omega)$ are denoted the spaces of generalized functions over $\mathcal{D}, \mathcal{D}(\Omega)$; $C^\infty(\Omega)$ is the space of infinitely differentiable functions in Ω.

2. Formulation of the main results. We shall consider the system

$$\mathcal{P}(x; D)u(x) = f(x); \tag{1}$$

here $u(x) = \{u_1(x), \dots, u_m(x)\}$, $f(x) = \{f_1(x), \dots, f_m(x)\}$ are column vectors of height m, and $\mathcal{P}(x; D) = \|P_{ij}(x; D)\|_{i,j=1,\dots,m}$ is a square matrix of linear differential operators with infinitely differentiable coefficients.

Suppose that the following conditions are satisfied:

A. At each fixed point $x \in \Omega$ the polynomial $Q(x;\xi) = \det \mathcal{P}(x;\xi)$ is hypoelliptic (in the sense of ([1])).

B. There exists a constant $C > 0$ such that for all $x', x'' \in \Omega$

$$\widetilde{Q}(x';\xi) < C\widetilde{Q}(x'';\xi).$$

* Somewhat more general hypoelliptic operators are contained in papers ([4–6]).

C. There exist nonnegative numbers $s_1, \dots, s_m$, $t_1, \dots, t_m$, and $\sigma > 0$ such that

$$\left|P_{ij}^{(\alpha)}(x;\xi)\right| < C[\widetilde{Q}(x;\xi)]^{t_j - s_i}(1 + |\xi|)^{-\sigma|\alpha|},$$

where

$$\sum_{i=1}^{m}(t_i - s_i) = 1$$

and $P_{ij} \equiv 0$ if $t_j - s_i < 0$.

We note that A and B are conditions of formal hypoellipticity of $\det \mathcal{P}$, while condition C is an analogue of the "nondegeneracy" condition ([7]). As the example in ([7]) (p. 4) shows, conditions A and B are not sufficient for hypoellipticity of the system (1).

Main theorem. *Let the system (1) in the domain Ω satisfy conditions A, B, C. Let $u \in \mathcal{D}'(\Omega)$ and $f \in C^\infty(\Omega)$. Then $u \in C^\infty(\Omega)$; in other words, the system (1) is hypoelliptic.*

We shall prove this theorem according to the scheme proposed in ([8]) (for the case of a single equation). As usual, the center of gravity will lie in the derivation of a priori estimates.

3. Some spaces of generalized functions. Denote by U^l the space of generalized vector-functions $u = \{u_1, \dots, u_m\}$, $u_j \in \mathcal{D}'$, for which the Fourier transform $\hat{u}(\xi)$ is an ordinary locally integrable function and

$$\|u, U^l\| = \left[\sum_{j=1}^{m} |\hat{u}_j(\xi)|^2 (1 + |\xi|)^{2l} \widetilde{Q}(\xi)^{2t_j}\, d\xi\right]^{1/2} < \infty. \tag{2}$$

Here $Q(\xi) = Q(0;\xi) = \det \mathcal{P}(0;\xi)$ $(0 \in \Omega)$. Similarly we define the space $\mathcal{F}^l$, consisting of vector-functions $f = \{f_1, \dots, f_m\}$, $f_j \in \mathcal{D}'$, for which

$$\|f, \mathcal{F}^l\| = \left[\sum_{i=1}^{m} |\hat{f}_i(\xi)|^2 (1 + |\xi|)^{2l} Q(\xi)^{2s_i}\, d\xi\right]^{1/2} < \infty. \tag{3}$$

Lemma 1. *For any l, $u \in U^l$ ($f \in \mathcal{F}^l$) if and only if $u \in U^{l-1}$ ($f \in \mathcal{F}^{l-1}$), and the expressions*

$$|h|^{-1}\|u_h - u, U^l\|, \qquad |h|^{-1}\|f_h - f, \mathcal{F}^l\|$$

are uniformly bounded as $|h| \to 0$. Here h is a vector in R^n; u_h, f_h are the translates of the generalized functions u, f by this vector.

Lemma 2. *Let $a \in \mathcal{D}$, $f \in \mathcal{F}^l$. Then $af \in \mathcal{F}^l$ and the estimate*

$$\|au, \mathcal{F}^l\| \leq \sup|a(x)|\,\|f, \mathcal{F}^l\| + C\|f, \mathcal{F}^{l-\gamma}\|, \tag{4}$$

$\gamma > 0$, holds; the constant $C > 0$ depends on $a(x)$, but does not depend on u.

The proof of this lemma is based on Lemma 3 and is analogous to the proof of Lemma 2 in ([8]).

Lemma 3. *Let $Q(\xi)$ be a hypoelliptic polynomial of order μ and let*

$$\mu(\xi) = (1 + |\xi|)^l[\widetilde{Q}(\xi)]^a, \qquad a > 0.$$

Then there exists $\sigma > 0$ such that

$$|\mu(\xi + \eta) - \mu(\xi)| < C(1 + |\eta|)^{a\gamma}(1 + |\xi|)^{-a\sigma}\mu(\xi).$$

This lemma is based on the fact that for every hypoelliptic polynomial $Q(\xi)$, for sufficiently large ξ, the inequality ([2])

$$|Q^{(\alpha)}(\xi)| < C|\xi|^{-\sigma|\alpha|}|Q(\xi)| \tag{5}$$

holds.

4. **The fundamental inequality.** The properties B, C of the operator (1) make it possible to write it in the form (cf. (2, 3))

$$\mathcal{P}(x; D) = \mathcal{P}(D) + \sum_{\omega} a_{\omega}(x)\mathcal{P}_{\omega}(D), \tag{6}$$

where $a_{\omega}(x) \in C^{\infty}$, $a_{\omega}(0) = 0$, $\mathcal{P}(D) = \mathcal{P}(0; D)$.

If $\mathcal{P}_{\omega}(D) = \|P_{\omega ij}(D)\|$, then, by virtue of C:

$$\left|P^{(\alpha)}_{\omega ij}(\xi)\right| \leq C(1 + |\xi|)^{-\sigma|\alpha|}\widetilde{Q}(\xi). \tag{7}$$

Since our aim is to prove the local regularity of solutions of (1), we may assume that $a_\omega(x) \in \mathcal{D}$ and that the quantity δ

$$\delta = \sum_\omega \sup |a_\omega(x)|$$

is sufficiently small. Under these assumptions we shall establish an estimate.

Theorem 1. *Let δ be sufficiently small, $u \in U^l$, $l' < l$. Then there exists a constant $C > 0$ (independent of u) such that*

$$\|u, U^l\| \leq C(\|\mathcal{P}(x;D)u, \mathcal{F}^{l'}\| + \|u, U^{l'}\|). \tag{8}$$

Proof. This theorem must be established for constant coefficients ($\mathcal{P}(x;D) = \mathcal{P}(D)$). The standard passage to variable coefficients is based on representation (6) and Lemma 2. Since smooth finite functions are dense in U^l, we may assume that $u_j \in \mathcal{D}$, $j = 1, \dots, m$. Thus, let $\mathcal{P}(D)u = f$. Passing to the Fourier transform, we obtain the algebraic system

$$\sum P_{jk}\hat{u}_k(\xi) = \hat{f}_j(\xi), \qquad j = 1, \dots, m,$$

from which it follows that

$$Q(\xi)u_i(\xi) = \sum_{j=1}^m P^{ji}(\xi)\hat{f}_j(\xi), \tag{9}$$

where $\|P^{ij}(\xi)\|$ is the matrix of algebraic complements of the matrix $\mathcal{P} = \|P_{ij}(\xi)\|$. By property C, $|P^{ji}(\xi)| \leq C\widetilde{Q}(\xi)^{1-t_j+s_i}$. Since the polynomial $Q(\xi)$ is hypoelliptic (i.e. (5) is satisfied), for $l' < l$

$$(1+|\xi|)^l\widetilde{Q}(\xi) \leq C\big[(1+|\xi|)^{l'}|Q(\xi)| + (1+|\xi|)^{l'}\widetilde{Q}(\xi)\big].$$

From these estimates and equality (9) we obtain inequality (8) for $\mathcal{P}(x;D) = \mathcal{P}(D)$.

5. **Theorem 2** (on the regularity of solutions in the whole space). *Let $u \in U^\lambda$ and $\mathcal{P}(x;D)u = f \in \mathcal{F}^l$. Then $u \in U^l$.*

Proof. Let $\lambda \leq l-1$. We shall show that $u \in U^{\lambda+1}$. By Lemma 1 it suffices for us to establish the uniform boundedness (in h) of the norms $\|\Delta_{hu}, U^\lambda\|$, where $\Delta_{hu} = |h|^{-1}(u_h - u)$. The function Δ_{hu} will satisfy the system

$$\mathcal{P}(x;D)\Delta_{hu} = \Delta_{hf} + \sum \Delta_{-h}a_\omega\mathcal{P}_\omega(D)u_h. \tag{10}$$

Applying Theorem 1 with $l = \lambda,\ l' = \lambda - 1$, we establish the uniform boundedness of $\|\Delta_{hu}, U^\lambda\|$, and together with it also that $u \in U^{\lambda+1}$. If $l - 1 < \lambda < l$, then, applying Theorem 1 with $l = l - 1,\ l' = \lambda - 1$, by means of (10) we at once establish the uniform boundedness of $\|\Delta_{hu}, U^l\|$, i.e. that $u \in U^l$. The theorem is proved.

6. Local regularity of solutions.

Let $u = \{u_1, \dots, u_m\}$, $u_j \in \mathcal{D}'(\Omega)$, where Ω is a bounded domain. We shall say that $u \in U^l_{\text{loc}}(\Omega)$ if $\varphi u \in U^l$ for every function $\varphi(x) \in \mathcal{D}$. The space $F^l_{\text{loc}}(\Omega)$ is defined analogously.

Lemma 4. Let α be a multi-index and $|\alpha| > 0$. If $u \in U^l_{\text{loc}}(\Omega)$, then $\mathcal{P}^{(\alpha)}(x; D)u \in F^{l+\sigma}_{\text{loc}}(\Omega)$.

Proof. If $v \in U^l$, then, by virtue of (7), $\mathcal{P}^{(\alpha)}_\omega(D)v \in U^{l+\sigma|\alpha|}$. Using (6) and Lemma 2, we find that $\mathcal{P}^{(\alpha)}(x; D)v \in U^{l+\sigma}$. Now let $u \in U^l_{\text{loc}}(\Omega)$, $\varphi \in \mathcal{D}(\Omega)$. Choose a function $\psi \in \mathcal{D}(\Omega)$ so that $\psi(x) = 1$ for $x \in \sup p\, \varphi$. Then

$$\varphi \mathcal{P}^{(\alpha)}(x; D)u = \varphi \mathcal{P}^{(\alpha)}(x; D)\psi u \in U^{l+\sigma}.$$

The lemma is proved.

Theorem 3. Let $u = \{u_1, \dots, u_m\}$, $u_j \in \mathcal{D}'(\Omega)$; let $\mathcal{P}(x; D)u \in F^l_{\text{loc}}(\Omega)$. Then $u \in U^l_{\text{loc}}(\Omega)$.

Proof. Let Ω' be any compact subdomain of Ω. Then there is such a λ that $u \in U^\lambda_{\text{loc}}(\Omega)$. We shall show that if $l \geq \lambda + \sigma$, then $u \in U^{\lambda+\sigma}_{\text{loc}}(\Omega)$. Indeed, by Leibniz' formula, for $\varphi \in \mathcal{D}(\Omega)$,

$$\mathcal{P}(x; D)(\varphi u) = \varphi \mathcal{P}(x; D)u + \sum D^\alpha \varphi\, \mathcal{P}^{(\alpha)}(x; D)/|\alpha|! \tag{11}$$

From Lemma 4 it follows that the right-hand side of (11) belongs to $U^{\lambda+\sigma}$. Then, according to Theorem 2, $\varphi u \in U^{\lambda+\sigma}$, i.e. $u \in U^{\lambda+\sigma}_{\text{loc}}(\Omega')$. Repeating these arguments, we find that $\varphi u \in U^l$, i.e. $u \in U^l_{\text{loc}}(\Omega')$, and in view of the arbitrariness of Ω', $u \in U^l_{\text{loc}}(\Omega)$. The theorem is proved.

The main theorem formulated above follows immediately from Theorem 3.

Received
21 I 1964

References

[1] L. Hörmander, Acta Math., **94**, 161 (1955).
[2] L. Hörmander, Comm. Pure and Appl. Math., **9**, 197 (1958).
[3] B. Malgrange, Bull. Soc. Math. France, **85**, 283 (1957).
[4] F. Trèves, Ann. Inst. Fourier, **9**, 1 (1959).
[5] F. Trèves, Am. J. Math., **83**, 645 (1961).

[6] L. Hörmander, Ann. Inst. Fourier, **11**, 477 (1961).
[7] L. R. Volevich, Matem. sborn., **59** (101) (supplementary), 3 (1962).
[8] J. Peetre, Comm. Pure and Appl. Math., **14**, 737 (1961).

Note: Figure translations are in progress. See original paper for figures.