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# L. A. SIMONYAN

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## Abstract

## Full Text

L. A. SIMONYAN

# ON TWO RADICALS OF LIE ALGEBRAS

(Presented by Academician A. I. Mal' tsev on 17 II 1964)

Throughout, by an "algebra" we mean a Lie algebra of infinite (not necessarily finite) dimension over a field of characteristic 0. A subalgebra of an algebra  $\Omega$  is called **subinvariant** if it is a member of some ascending normal series in  $\Omega$ . An element of the algebra  $\Omega$  is called **subinvariant** if the one-dimensional subalgebra generated by it is subinvariant in  $\Omega$ . An algebra each element of which is subinvariant is called a  $B$ -algebra. A subalgebra of an algebra  $\Omega$  that is a member of some finite normal series in  $\Omega$  is called **attainable**. An element is **attainable** if the one-dimensional subalgebra generated by it is attainable. An algebra each element of which is attainable is called an  $SB$ -algebra. The purpose of the present note is to define the  $B$ -radical of an algebra  $\Omega$ , the subalgebra in  $\Omega$  generated by all its  $B$ -ideals (a  $B$ -ideal is an ideal that is a  $B$ -algebra), and the  $SB$ -radical. In connection with this it is proved that the sum of all  $B$ -ideals of an algebra is again a  $B$ -ideal, and the sum of all  $SB$ -ideals of an algebra is again an  $SB$ -ideal. The latter is a transfer to the case of Lie algebras of the following theorem proved in a paper of R. Baer <sup>(1)</sup>.

*The product of normal divisors that are nilgroups is again a nilgroup.*

Let  $\Omega$  be an algebra and  $D$  a derivation of the algebra  $\Omega$ . We introduce the notation:  $xD^0 = x$ ,  $xD$  is the image of the element  $x$  under this derivation, and  $xD^n = (xD^{n-1})D$ .  $D$  is called a **nilderivation** if for every  $x \in \Omega$  there exists an  $n$  such that  $xD^n = 0$ .

**Lemma 1.** *If  $D$  is a nilderivation of the algebra  $\Omega$ , then  $\exp D =$*

$$= \sum_{i=0}^{\infty} \frac{D^i}{i!}$$

*is an automorphism of the algebra  $\Omega$ .*

An element  $y$  of the algebra  $\Omega$  is called a **nil element** if  $\text{ad } y$  is a nilderivation. An algebra all of whose elements are nil elements is called a **nilalgebra**.

**Lemma 2.** *A nil element  $x$  belongs to the normalizer of a subalgebra  $\mathfrak{M}$  of the algebra  $\Omega$  if and only if  $\mathfrak{M} \exp(\text{ad } x) = \mathfrak{M}$ .*

By  $LN(\Omega)$  we shall denote the **locally nilpotent radical** of the algebra  $\Omega$  <sup>(2)</sup>.

**Lemma 3.** *If  $\mathfrak{M}$  is an ideal of a nilalgebra  $\Omega$ , then  $LN(\mathfrak{M}) \subseteq LN(\Omega)$ .*

**Lemma 4.** *Let the algebra  $\Omega$  possess a local system of subalgebras  $\{\Omega_\alpha\}$ , and in each subalgebra  $\Omega_\alpha$ , moreover, let an ideal  $\mathfrak{M}_\alpha$  be chosen, with  $\Omega_\alpha \subset \Omega_\beta$  always implying  $\mathfrak{M}_\alpha \subseteq \mathfrak{M}_\beta$ . Denote by  $\mathfrak{M}$  the set-theoretic sum of all  $\mathfrak{M}_\alpha$ . Then  $\mathfrak{M}$  is an ideal in  $\Omega$  with local system  $\{\mathfrak{M}_\alpha\}$ .*

With the aid of Lemmas 3 and 4 one proves:

**Theorem 1.** *Every B-algebra is locally nilpotent.*

**Lemma 5.** *If  $\mathfrak{M}$  is a subinvariant subalgebra of a locally finite algebra  $\Omega$ , and an element  $x$  from the normalizer of the subalgebra  $\mathfrak{M}$  is subinvariant in  $\Omega$ , then  $\{\mathfrak{M}, x\}$  is subinvariant in  $\Omega$ .*

Let

$$\mathfrak{M} = \mathfrak{M}_0 \subset \mathfrak{M}_1 \subset \dots \subset \mathfrak{M}_\alpha \subset \mathfrak{M}_{\alpha+1} \subset \dots \subset \mathfrak{M}_\gamma = \Omega$$

an increasing normal series in  $\Omega$ . We show that the subalgebras

$$\mathfrak{N}_\alpha = \bigcap_{n=-\infty}^{\infty} \mathfrak{M}_\alpha \exp(n \operatorname{ad} x)$$

form in  $\Omega$  an increasing normal series, possibly with repetitions, whose terms are invariant with respect to  $x$ . From Lemma 2 it follows that  $\mathfrak{N}_0 = \mathfrak{M}$ ,  $\mathfrak{N}_\gamma = \Omega$ , and  $[\mathfrak{N}_\alpha, x] \subseteq \mathfrak{N}_\alpha$ . It is clear that  $\mathfrak{N}_\alpha \subseteq \mathfrak{N}_{\alpha+1}$ . Further, since the subalgebra  $\mathfrak{N}_\alpha \exp(n \operatorname{ad} x)$  is invariant in the subalgebra  $\mathfrak{N}_{\alpha+1} \exp(n \operatorname{ad} x)$ , it follows that  $\mathfrak{N}_{\alpha+1}$  belongs to the normalizer of this subalgebra, and therefore  $\mathfrak{N}_\alpha$  is an ideal in  $\mathfrak{N}_{\alpha+1}$ . It remains to show that if  $\beta < \gamma$  is a limit number, then

$$\mathfrak{N}_\beta = \bigcup_{\alpha < \beta} \mathfrak{N}_\alpha.$$

Let  $y \in \mathfrak{N}_\beta$ . This means that for every  $n$  there is a  $y_n$  in  $\mathfrak{M}_\beta$  such that  $y = y_n \exp(n \operatorname{ad} x)$ . The latter is equivalent to saying that all  $y \exp(-n \operatorname{ad} x)$  are contained in  $\mathfrak{M}_\beta$ . The subalgebra generated by all  $y \exp(-n \operatorname{ad} x)$  will be contained in the finite-dimensional subalgebra generated by the elements  $x$  and  $y$ . Therefore there exists an  $\alpha < \beta$  such that, for all  $n$ ,  $y \exp(-n \operatorname{ad} x)$  is contained in  $\mathfrak{M}_\alpha$ . Consequently,  $y \in \mathfrak{N}_\alpha$  and

$$\mathfrak{N}_\beta = \bigcup_{\alpha < \beta} \mathfrak{N}_\alpha.$$

We now show that  $\{\mathfrak{M}, x\}$  is subinvariant in  $\Omega$ . Since  $x$  is subinvariant in  $\{\mathfrak{N}_{\alpha+1}, x\}$  and  $\mathfrak{N}_\alpha$  is an ideal in  $\{\mathfrak{N}_{\alpha+1}, x\}$ , there exists in  $\{\mathfrak{N}_{\alpha+1}, x\}$  an increasing normal series whose first term is  $\{\mathfrak{N}_\alpha, x\}$ . It is easy to verify that the terms

of all such series form in  $\Omega$  an increasing normal series, possibly with repetitions. Consequently, the subalgebra  $\{\mathfrak{M}, x\}$  is subinvariant in  $\Omega$ .

**Lemma 6.** *Every subalgebra with a finite number of generators of a  $B$ -algebra is subinvariant in it.*

Let  $\Omega$  be a  $B$ -algebra and  $\mathfrak{M}$  its subalgebra with a finite number of generators. By Theorem 1,  $\mathfrak{M}$  is a finite-dimensional nilpotent algebra. Consequently, in  $\mathfrak{M}$  one can construct a finite normal series with one-dimensional factors. From Lemma 5, by induction on the length of this series, it follows that  $\mathfrak{M}$  is subinvariant in  $\Omega$ .

**Lemma 7.** *If an ideal  $\mathfrak{M}$  of an algebra  $\Omega$  is a  $B$ -algebra and if there exists a subinvariant element  $x$  in  $\Omega$  such that  $\{\mathfrak{M}, x\} = \Omega$ , then  $\Omega$  is also a  $B$ -algebra.*

Let  $y \in \mathfrak{M}$ . Since  $\mathfrak{M}$  is locally nilpotent (Theorem 1),  $\Omega$  is also locally nilpotent<sup>(2)</sup>. The subalgebra  $\mathfrak{N}$ , generated by the elements  $x$  and  $y$ , is a finite-dimensional nilpotent subalgebra. Consequently,  $\mathfrak{M} \cap \mathfrak{N}$  is subinvariant in  $\Omega$  (Lemma 6). Since  $\mathfrak{M} \cap \mathfrak{N}$  is an ideal in  $\mathfrak{N}$  and  $\{\mathfrak{M} \cap \mathfrak{N}, x\} = \mathfrak{N}$ , by Lemma 5,  $\mathfrak{N}$  is subinvariant in  $\Omega$ . From the fact that  $x + y$  is subinvariant in  $\mathfrak{N}$ , it now follows that  $x + y$  is subinvariant in  $\Omega$ .

**Theorem 2.** *The sum of all  $B$ -ideals of an algebra is again a  $B$ -ideal.*

Let  $\Omega$  be an algebra, let  $\mathfrak{M}$  and  $\mathfrak{N}$  be its  $B$ -ideals, and let  $x \in \mathfrak{M} + \mathfrak{N}$ . The element  $x$  can be represented in the form  $x = y + z$ , where  $y \in \mathfrak{M}$  and  $z \in \mathfrak{N}$ . Since  $y$  is subinvariant in  $\Omega$ , the subalgebra  $\{\mathfrak{N}, y\}$  is subinvariant in  $\Omega$ . By Lemma 7,  $\{\mathfrak{N}, y\}$  is a  $B$ -algebra. From the fact that  $\{\mathfrak{N}, y\} = \{\mathfrak{N}, x\}$ , it now follows that  $x$  is subinvariant in  $\Omega$ .

By induction it is proved that the sum of any finite set of  $B$ -ideals is again a  $B$ -ideal. If  $B(\Omega)$  is the sum of all  $B$ -ideals of the algebra  $\Omega$  and  $x \in B(\Omega)$ , then  $x$  is contained in the sum of some finite number of  $B$ -ideals. Consequently,  $x$  is subinvariant in  $\Omega$ .

In a manner analogous to that set out above, one proves:

**Theorem 3.** *The sum of all  $SB$ -ideals of an algebra is again an  $SB$ -ideal.*

A derivation  $D$  of an algebra  $\Omega$  is called **stable** if in  $\Omega$  there exists an increasing normal series of subalgebras

$$0 = \Omega_0 \subset \Omega_1 \subset \dots \subset \Omega_\alpha \subset \Omega_{\alpha+1} \subset \dots \subset \Omega_\gamma = \Omega$$

such that  $xD \in \Omega_\alpha$  for every  $x \in \Omega_{\alpha+1}$ .

**Lemma 8.** *Every nil-derivation of a commutative algebra is stable.*

An algebra is called an  $RN^*$ -**algebra** if in it there exists an increasing normal series with commutative factors. The  $RN^*$ -**radical** of an algebra  $\Omega$  will be denoted by  $RN^*(\Omega)$ .

With the aid of Lemma 8 it is proved.

**Theorem 4.**  $B(\Omega) = LN(\Omega) \cap RN^*(\Omega)$ .

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## REFERENCES

<sup>1</sup> R. Baer, Math. Zs., **62**, 4, 402 (1955). <sup>2</sup> B. I. Plotkin, UMN, **13**, 6, 133 (1958).

*Note: Figure translations are in progress. See original paper for figures.*

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