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Abstract

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AN EXACT METHOD FOR CALCULATING A PERIODIC CELLULAR WAVEGUIDE

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The calculation of a periodic cellular waveguide is reduced to determining, from the given geometrical dimensions of the cell and the frequency ω, the wave numbers ψ_l and the forms $\left\|\begin{matrix} F_1^{(l)} \\ F_2^{(l)} \end{matrix}\right\|$ of the normal waves traveling along the chain of cells $j = 1, 2, \ldots$

$$\left\|\begin{matrix} F_1^{(l)}(q_1) \\ F_2^{(l)}(q_2) \end{matrix}\right\| \exp\left[i(\omega t - \psi_l j)\right], \tag{1}$$

where $F_1^{(l)}(q_1)$ and $F_2^{(l)}(q_2)$ are distribution functions of the tangential components of the field E_τ and H_τ, specified in some section S_1 of the cell, for example at the input aperture $S_1 = S_{\text{in}}$, i.e. $F_1 = E_\tau(q_1)$, $F_2 = H_\tau(q_1)$, $q_1 \in S_1$, or else any other pair of functions $\{E_\tau(q_1),\ H_\tau(q_2)\}$, $\{E_\tau(q_1),\ E_\tau(q_2)\}$, $\{H_\tau(q_1),\ H_\tau(q_2)\}$, specified in two different sections of the cell S_1 and S_2 ($q_1 \in S_1,\ q_2 \in S_2$), and in one-to-one correspondence with $\{E_\tau(q_1),\ H_\tau(q_1)\}$, $q_1 \in S_1$. The method proposed here for calculating the parameters of the normal waves of the first numbers l is based on the work (^{1}). It consists in solving the eigenvalue equation $\lambda_l = \exp(-i\psi_l)$ for the Breisig operator of the cell

$$A\left\|\begin{matrix} F_1 \\ F_2 \end{matrix}\right\| = \lambda \left\|\begin{matrix} F_1 \\ F_2 \end{matrix}\right\|; \qquad A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}. \tag{2}$$

With the aid of a certain system of basis functions $\{F_{1s}\}_1^\infty$ and $\{F_{2s}\}_1^\infty$, introduced in the sections S_1 and S_2, the functional operators A_{ik} are replaced by matrices A_{ik}, and (2) becomes a homogeneous infinite system of algebraic equations. The condition for its consistency is the characteristic equation of the Breisig operator

$$|A - \lambda \mathcal{E}| = 0 \tag{3}$$

with roots ψ_l (the so-called dispersion equation of the waveguide). Replacing the infinite matrices A_{ik} by finite matrices A_{ik}^N of order N, we find approximate

Fig. 1

Figure 1: Fig. 1

values ψ_l^N. To improve the convergence of ψ_l^N to ψ_l as $N \to \infty$, electrostatic singularities of the field E on the sharp edges of the inner surface of the cell are taken into account in $\{F_{1s}\}_1^\infty$ and $\{F_{2s}\}_1^\infty$. We demonstrate this method on the classical example of axially symmetric waves of a circular iris-loaded waveguide with the cell shown in Fig. 1 ($^{2-6}$).

Fig. 1

We construct the matrix representation (2) directly from the boundary-value problem for Maxwell' s equations, taking as the basis system the functions

$$E_r\left(r,\ z = \pm\frac{t}{2}\right) = \sum_{s=1}^{\infty} K_s^{(1,2)}\, \mathcal{E}_s(r), \quad 0 < r < a, \tag{4}$$

which determine the electric fields in sections $\langle 2\rangle$ and $\langle 4\rangle$ (Fig. 1). Here $\{\mathcal{E}_s(r)\}_{s=1}^\infty$ is an arbitrary complete system of functions. By virtue of the first problem

of electrodynamics (7), specifying the fields E_r in sections $\langle 2\rangle$ and $\langle 4\rangle$ is equivalent to specifying the fields E_r and H_φ in the input or output sections of the cell. Applying the method of partial regions (see, for example, (^{8})), we represent the wave fields (1) E_r and H_φ in the regions $p = 1, 2, 3$ as sums of normal waves traveling along the z-axis of the sleeves $p = 1, 2, 3$ (Fig. 1):

$$E_r^{(p)} = \sum_{n=1}^{\infty} e_n^{(p)}(r) \left[C_{n1}^{(p)} e^{-\gamma_n^{(p)} z^{(p)}} - C_{n2}^{(p)} e^{\gamma_n^{(p)} z^{(p)}}\right] \frac{\gamma_n^{(p)}}{i\omega\varepsilon}, \tag{5}$$

$$H_\varphi^{(p)} = \sum_{n=1}^{\infty} e_n^{(p)}(r) \left[C_{n1}^{(p)} e^{-\gamma_n^{(p)} z^{(p)}} + C_{n2}^{(p)} e^{\gamma_n^{(p)} z^{(p)}}\right], \tag{6}$$

where p is the number of the sleeve (region); n is the number of the wave in the sleeve; $\gamma_n^{(p)} = \sqrt{(\rho_{0n}/r_p)^2 - (\omega/c)^2}$ are the wave numbers of these waves; ρ_{0n} is the root of order n of the Bessel function $J_0(\rho)$; $z^{(1)} = z + t/2$; $z^{(2)} = z$; $z^{(3)} = z - t/2$; $r_1 = r_3 = b$; $r_2 = a$; $c = 3 \cdot 10^{10}$ cm/sec; ε is the dielectric constant. The functions $e_n^{(p)}(r) = J_1(\rho_{0n} r/r_p)/\sqrt{\pi r_p}\,|J_1(\rho_{0n})|$ are normalized so that the scalar products $(e_n^{(p)}, e_m^{(p)}) = \delta_{n,m}$. By virtue of the definition of a normal wave (1) in a cellular waveguide,

$$C_{n1}^{(3)} = C_{n1}^{(1)} e^{\gamma_n d - i\psi}; \qquad C_{n2}^{(3)} = C_{n2}^{(1)} e^{-\gamma_n d - i\psi}; \qquad \gamma_n = \gamma_n^{(1)} = \gamma_n^{(3)} \tag{7}$$

and we are in fact dealing with a closed volume composed of the volume V_{1+3} of sleeves 1 and 3 and the volume V_2 of sleeve 2.

Let us solve the first problem of electrodynamics (^{7}) for V_2 for the prescribed fields (4). We expand (4) in the functions $e_n^{(1)} = e_n^{(3)}$:

$$E_r(r, z = \mp t/2) = \sum_{s=1}^{\infty} \sum_{n=1}^{\infty} K_s^{(1,2)} \beta_{ns} e_n^{(1)}, \tag{8}$$

where $\beta_{ns} = (\mathcal{E}_s, e_n^{(1)})$ are Fourier coefficients; $\tilde{\mathcal{E}}_s = \mathcal{E}_s$ for $0 < r < a$ and $\tilde{\mathcal{E}}_s = 0$ for $a < r < b$.

Having determined E_r in the sections $z = \mp t/2$ and $z = \mp D/2$ from (5), and equating E_r at $z = \mp t/2$ to expression (8), and for E_r at $z = \mp D/2$ taking (7) into account, we obtain

$$C_{n2}^{(1)} = \frac{i\omega\varepsilon}{2\gamma_n \operatorname{sh} \gamma_n d} \left(e^{i\psi} \sum_{s=1}^{\infty} \beta_{ns} K_s^{(2)} - e^{\mp\gamma_n d} \sum_{s=1}^{\infty} \beta_{ns} K_s^{(1)} \right). \tag{9}$$

Solving the first problem of electrodynamics for V_2, excited at $z = \mp t/2$ by the fields (4), we obtain

$$C_{n2}^{(2)} = \frac{i\omega\varepsilon}{2\alpha_n \operatorname{sh} \alpha_n t} \left(e^{\pm\alpha_n t/2} \sum_{s=1}^{\infty} \alpha_{ns} K_s^{(1)} - e^{\mp\alpha_n t/2} \sum_{s=1}^{\infty} \alpha_{ns} K_s^{(2)} \right), \tag{10}$$

where $\alpha_{ns} = (\mathcal{E}_s, e_n^{(2)})$, $\alpha_n = \gamma_n^{(2)}$.

The continuity condition for the fields H_φ in the apertures $0 < r < a$, $z = \mp t/2$, will be

$$\sum_{n=1}^{\infty} e_n^{(1)} \left(C_{n1}^{(1,3)} + C_{n2}^{(1,3)} \right) = \sum_{n=1}^{\infty} e_n^{(2)} \left(C_{n1}^{(2)} e^{\pm\alpha_n t/2} + C_{n2}^{(2)} e^{\mp\alpha_n t/2} \right). \tag{11}$$

Multiplying (11) scalarly by $\mathcal{E}_{s'}(r)$, $s' = 1, 2, \ldots$, we replace (11) by a system of algebraic equations

$$\sum_{n=1}^{\infty} \beta_{ns'} \left(C_{n1}^{(1,3)} + C_{n2}^{(1,3)} \right) = \sum_{n=1}^{\infty} \alpha_{ns'} \left(C_{n1}^{(2)} e^{\pm\alpha_n t/2} + C_{n2}^{(2)} e^{\mp\alpha_n t/2} \right), \tag{11'}$$

$$s' = 1, 2, \ldots, \infty.$$

Substituting (9) and (10) into (11′) and taking (7) into account, we obtain an infinite homogeneous system of algebraic equations for determining K_s. We write it in operator form, introducing the vectors $\overline{K}^{(1)}$ and $\overline{K}^{(2)}$.

$$\begin{pmatrix} a-\alpha & \alpha \\ \alpha & a \end{pmatrix} \left\| \begin{matrix} \overline{K}^{(1)} \\ \overline{K}^{(2)} \end{matrix} \right\| = \begin{pmatrix} 0 & e^{i\psi}\beta \\ e^{-i\psi}\beta & 0 \end{pmatrix} \left\| \begin{matrix} \overline{K}^{(1)} \\ \overline{K}^{(2)} \end{matrix} \right\|, \tag{12}$$

where a, α, and β are infinite symmetric square matrices with elements

$$a_{ss'} = \sum_{n=1}^{\infty} \frac{\operatorname{cth}\gamma_n d}{\gamma_n} \beta_{ns}\beta_{ns'} + \sum_{n=1}^{\infty} \frac{\operatorname{cth}\alpha_n t}{\alpha_n} \alpha_{ns}\alpha_{ns'}, \tag{13}$$

$$\alpha_{ss'} = \sum_{n=1}^{\infty} \frac{\alpha_{ns}\alpha_{ns'}}{\alpha_n \operatorname{sh}\alpha_n t}, \qquad \beta_{ss'} = \sum_{n=1}^{\infty} \frac{\beta_{ns}\beta_{ns'}}{\gamma_n \operatorname{sh}\gamma_n d},$$

where α_{ns} and β_{ns} are determined from (8) and (10). In the case when the electrostatic singularity of the field E_r at the edges of the apertures in the cross sections ⟨3⟩ and ⟨4⟩ is taken into account in the functions $\mathcal{E}_s$ in the form $\mathcal{E}_s = (r/a)^{2s-1}/\sqrt{1-(r/a)^2}$, they are expressed by the formulas

$$\alpha_{ns} = \frac{2\sqrt{\pi}}{|J_1(\rho_{0n})|} \sum_{r=0}^{s-1} (-1)^r \, {}_{s-1}C_r \, (2r-1)!! \, (\rho_{0n})^{-(r+1)} I_{r+3/2}(\rho_{0n}); \tag{14}$$

$$\beta_{ns} = \frac{a \cdot 2\sqrt{\pi}}{b\,|J_1(\rho_{0n})|} \sum_{r=0}^{s-1} (-1)^r \, {}_{s-1}C_r \, (2r-1)!! \left(\rho_{0n}\frac{a}{b}\right)^{-(r+1)} I_{r+3/2}\left(\rho_{0n}\frac{a}{b}\right),$$

where ${}_{s-1}C_r$ are binomial coefficients; $I_q(x) = I'_q(x)/\sqrt{\pi x/2}$ are Bessel functions of order q. Equation (12) is the required eigenvalue equation (3) for the Breizig operator (2) in operator form in the basis of functions (4) specified on the internal cross sections of the cell ⟨2⟩ and ⟨4⟩. For numerical calculations of ψ in the passband ($|\cos\psi| < 1$), using the substitution $\overline{K}^{(1,2)} = \overline{X} \pm i\overline{Y}$, it is convenient to represent it in the form

$$\begin{pmatrix} a-\alpha-\cos\psi\,\beta & -\sin\psi\,\beta \\ -\sin\psi\,\beta & a+\alpha+\cos\psi\,\beta \end{pmatrix} \left\| \begin{matrix} \overline{X} \\ \overline{Y} \end{matrix} \right\| = 0. \tag{15}$$

Then the dispersion equation in matrix form takes the form

$$\det \begin{pmatrix} a-\alpha-\beta\cos\psi & -\beta\sin\psi \\ -\beta\sin\psi & a+\alpha+\beta\cos\psi \end{pmatrix} = 0. \tag{16}$$

For the numerical solution of (16), the matrix blocks in (16) were truncated at the N-th row and N-th column. For $N = 1$ and $t \to 0$, (16) coincides with the equation obtained in work (^{9}), and for $N = 1$ and finite t, with the equation of work (^{10}).

Table 1 gives the results of calculations of ψ_1^N (in radians) on the BESM-2 by equation (16) for the fundamental passband ($l = 1$) at various frequencies ω, determined by the free-space wavelength λ_0. The calculation is given for a cell with parameters (see Fig. 1) $a/b = 0.30000$, $b = 4.30000$, $t = 0.40000$, $d = 1.20200$. As is seen from the table, ψ_1^N ceases to change with increasing N in the second decimal place for $N \geqslant 3$, which ensures an accuracy of the order of 10–20 min. We note that for $\lambda_0 = 10.677$ cm, for a cell with the dimensions indicated above, experiment gives $\psi_1 = \pi/2$ with an accuracy of up to several tens of minutes, whereas the calculated value is $\psi_1 = 1.571$ (for $N = 3$), and for a cell with $a/b = 0.5$, $b = 5.525$, $t = 0.4$ and $d = 0.778$, at $\lambda = 11.039$ cm, $\psi_{\text{expt}} = \pi/2$, while $\psi_{\text{calc}} = 1.577$ ($N = 4$). The experimental data were taken from work (10); since their accuracy is lower than the calculated one, this comparison does not make it possible to assess the accuracy of the calculation.

Table 1*

λ_0, cm	$N = 1$	$N = 2$	$N = 3$	$N = 4$
10.4	–	–	–	–
10.5	–	2.8657	2.8155	2.8070
10.6	2.0646	1.9740	1.9635	1.9610
10.7	1.5410	1.4740	1.4690	1.4665
10.8	1.0750	1.0230	1.0195	1.0180
10.9	0.5100	0.4670	0.4650	0.4631
11.0	–	–	–	–

* The imaginary values of ψ_1 are marked with a dash.

Increasing the accuracy of the calculation of ψ_1 by increasing $N > 4$ on a machine of the BESM-2 type is impossible, since as N grows the matrices a, α and β tend to become singular, the determinant in (16) tends to 0, and for an accurate determination of the root ψ_1 one must compute the elements a, α and β with accuracy better than 10^{-8}. In Table 1 they were computed with accuracy 10^{-6}.

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REFERENCES

1. P. E. Krasnushkin, *Radiotekhn. i elektronika* (in press).

2. V. V. Vladimirskii, *DAN*, **52**, No. 3, 219 (1946); *ZhTF*, **17**, 11, 1277 (1947).

3. E. L. Chu, W. W. Hanson, *J. Appl. Phys.*, **18**, 996 (1947); **20**, 280 (1949).

4. L. Brillouin, *J. Appl. Phys.*, **19**, 1023 (1948).

5. W. Walkinshow, *Proc. Roy. Soc.*, **61**, 246 (1948); *J. Appl. Phys.*, **20**, 634 (1949).

6. R. Combe, M. Feix, *Nuovo Cim.*, **15**, No. 5, 760 (1960).

7. Ya. N. Fel' d, *Foundations of the Theory of Slotted Antennas*, Moscow, 1948.

8. G. V. Kisun' ko, *Electrodynamics of Hollow Systems*, Leningrad, 1949.

9. G. N. Rapoport, *ZhTF*, **21**, 9, 1076 (1951); **27**, 9, 2105 (1957).

10. A. G. Tragov, Candidate' s dissertation, Moscow Engineering-Physics Institute, 1962.

Note: Figure translations are in progress. See original paper for figures.

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