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EQUATION OF  
ARBITRARY ORDER  
WITH VARIABLE  
COEFFICIENTS,  
INCLUDING THE CASE  
OF DEGENERATION OF  
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ON THE BOUNDARY  
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**Abstract**

**Full Text**

**MATHEMATICS**

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**ON A BOUNDARY ESTIMATE FOR THE SOLUTION OF AN ELLIPTIC-TYPE EQUATION OF ARBITRARY ORDER WITH VARIABLE COEFFICIENTS, INCLUDING THE CASE OF DEGENERATION OF THE COEFFICIENTS ON THE BOUNDARY OF THE DOMAIN**

*(Presented by Academician G. I. Petrov, 20 X 1963)*

In the literature <sup>(1, 2)</sup> there are estimates of the growth of solutions of elliptic-type equations near the boundary of the domain in which they are prescribed, obtained in the metric  $C$  chiefly on the basis of studying the corresponding Green's function. S. M. Nikol'skii <sup>(3)</sup> obtained other estimates, sharp with respect to order, for the growth of a harmonic function and its derivatives near the boundary of a domain in the sense of  $L_p$ . In <sup>(4)</sup> sharp estimates close to those of <sup>(3)</sup> were obtained for the boundary of an  $n$ -dimensional domain for the solution of an elliptic-type equation of arbitrary order with constant coefficients in the  $L_2$  metric. The purpose of the present paper is to obtain similar estimates in the case of variable coefficients, including the case when they undergo degeneration on the boundary of the domain. The investigations are in fact carried out in the two-dimensional case; they carry over to the  $n$ -dimensional case by analogy.

§ 1. In the case of two dimensions, we consider a solution  $u$  of a differential equation which is the Euler equation for the functional (Dirichlet integral)

$$D_g^{(l)}(u) = \iint_g \left[ \sum_{i=0}^l a_i(x, y) \left( \frac{\partial^l u}{\partial x^{l-i} \partial y^i} \right)^2 + 2 \sum_{\substack{i,p=0 \\ i \neq p, b_{ip}=b_{pi}}}^l b_{ip}(x, y) \frac{\partial^l u}{\partial x^{l-i} \partial y^i} \frac{\partial^l u}{\partial x^{l-p} \partial y^p} \right] dx dy. \tag{1,1}$$

It is assumed that  $g$  is a bounded domain with boundary  $\Gamma$  of class  $C^{(l+q+2)}$ , where the integer  $l + q + 2$  is related to the highest order of derivative (of the function  $u$ ) which we shall estimate.

The  $a_i, b_{ip}$  satisfy the following conditions in  $g$ : 1)  $a_i, b_{ip} \in W_{2,-s}^{(s)}(g)$ ,  $s =$

1, 2, ..., q, q + 1, i.e. the integrals

$$\iint_g (D_s a_i)^2 t^{2s} dg \leq M < +\infty,$$

where  $M > 0$  is a constant;  $D_s a_i$  is any partial derivative of  $a_i$  of order  $s$ ;  $t$  is the distance from the point  $(x, y) \in g$  to the boundary  $\Gamma$  of the domain  $g$  in the normal direction; 2) for any real  $\xi_i, \xi_p$  the inequality holds

$$\sum_{i=1}^l a_i(x, y) \xi_i^2 + 2 \sum_{\substack{i,p=0 \\ i \neq p}}^l b_{ip}(x, y) \xi_i \xi_p \geq k \sum_{i=0}^l \xi_i^2 \rho^{2\alpha}(x, y); \quad (1,2)$$

$k > 0$ ,  $\alpha \geq 0$  are constants, independent of either  $\xi_i$  or the point  $(x, y)$ . The function  $\rho(x, y) \geq 0$  is defined as follows:

$$\rho(x, y) = \begin{cases} t & \text{on } \Pi_{0,\delta} = g - g_\delta, \\ 1 & \text{on } g_\delta \ (0 < \delta < 1), \end{cases} \quad (1,3)$$

where  $t \geq 0$  was defined above;  $g_\delta$  is the domain obtained from  $g$  by removing the strip  $0 \leq t \leq \delta$ . The number  $\alpha \geq 0$  will be called the **degree of degeneration of the functional (1, 1) on the boundary**.

§ 2. We construct (see (4)) an auxiliary function  $\lambda_m(t) \geq 0$ , defined on the half-axis  $0 \leq t < +\infty$  and satisfying the inequalities

$$\left| \lambda_m^{(s)}(t) \right|^{\frac{2(l+q+1+m)}{2(l+q+1+m)-s}} \leq A_m \lambda_m(t); \quad (2,1)$$

$$\left| \lambda_m^{(s)}(t) \right|^{\frac{2(l+q+1+m-s)}{2(l+q+1+m)-s}} \leq B_m (t - H)^{2(l+q+1+m-s)}; \quad (2,2)$$

$$(t - H)^{2(l+q+1+m)} \leq D_m \lambda_m(t), \quad (2,3)$$

where  $A_m, B_m, D_m$  are constants independent of  $H$  and  $t$ , and the parameter  $m \geq 0$  is otherwise arbitrary.

§ 3. Near the boundary  $\Gamma$  we introduce a new coordinate net  $(s, t)$ :  $t$  is the distance from the point to  $\Gamma$  in the direction of the inward normal,  $s$  is the length of the arc of  $\Gamma$ . Let  $g_\delta$  be the domain obtained by removing from  $g$  the strip  $0 \leq t \leq \delta$ . We construct the auxiliary function:

$$\eta(x, y) = \begin{cases} 0 & \text{on } g - g_H, \\ \lambda_m(t) & \text{on } g_H - g_\delta = \Pi_{H,\delta}, \\ 1 & \text{on } g_\delta. \end{cases} \quad (3,1)$$

On  $\Pi_{H,\delta}$  the inequality (see (4)) will hold

$$\left| \frac{\partial^p \eta}{\partial x^{p-i} \partial y^i} \right| \leq L \left| \lambda_m^{(p)}(t) \right|, \quad (3,2)$$

where  $L > 0$  is a constant;  $i = 0, 1, \dots, p$ ;  $p = 1, 2, \dots, l + q + 1$ .

§ 4. Let a function  $u(x, y)$  be defined on  $g$ , possessing generalized derivatives up to order  $l$  inclusive, and such that for all  $\sigma = H - gh > 0$

$$D_{g_\sigma}^{(l)}(u) < +\infty. \quad (4,1)$$

It follows from (4,1) and (1,2) that the function  $u(x, y)$  has generalized derivatives of order  $l$  with integrable square on  $g_\sigma$ . We impose further conditions on the function  $u$ , namely: whatever  $\sigma > 0$  may be, and for every function  $v$  with  $D_{g_\sigma}^{(l)}(v) < +\infty$  such that

$$\left. \frac{\partial^k v}{\partial n^k} \right|_{\Gamma_\sigma} = 0 \quad (k = 0, 1, \dots, l - 1) \quad (4,2)$$

(where  $\Gamma_\sigma$  is the boundary of the domain  $g_\sigma$ , and  $n$  is the inward normal), the equality

$$D_{g_\sigma}^{(l)}(u, v) = \iint_{g_\sigma} \left[ \sum_{i=0}^l a_i(x, y) \frac{\partial^l u}{\partial x^{l-i} \partial y^i} \frac{\partial^l v}{\partial x^{l-i} \partial y^i} + \sum_{\substack{i,p=0 \\ i \neq p}}^l b_{ip}(x, y) \left( \frac{\partial^l u}{\partial x^{l-i} \partial y^i} \frac{\partial^l v}{\partial x^{l-p} \partial y^p} + \frac{\partial^l u}{\partial x^{l-p} \partial y^p} \frac{\partial^l v}{\partial x^{l-i} \partial y^i} \right) \right] dx dy = 0. \quad (4,3)$$

Then  $u$  will be a generalized solution of the Euler equation corresponding to the functional (1,1).

§ 5. **Lemma 1.** *Let  $g$  be an arbitrary domain, and let a function  $\varphi(x, y)$  be given on  $g$ , having the property that it and all its partial derivatives up to order  $r - 1$  inclusive are square-integrable on  $g$ , and suppose it is known that, for any  $s = 0, 1, \dots, r$ , the integral of the finite difference with step  $h$*

$$\iint_{g_{rh}} \left| \frac{\Delta_{s,r-s}\varphi}{h^r} \right|^2 dx dy \leq M < +\infty \tag{5,1}$$

( $g_\sigma$  is the domain of points of  $g$  whose distance from the boundary  $\Gamma$  of the domain  $g$  is greater than  $\sigma$ ). Then: 1) on  $g$  there exist generalized partial derivatives

$\partial^r \varphi / \partial x^s \partial y^{r-s}$ ; 2) the integral

$$\iint_{g_*} \left| \frac{\Delta_{s,r-s}\varphi}{h^r} - \frac{\partial^r \varphi}{\partial x^s \partial y^{r-s}} \right|^2 d\sigma_* \rightarrow 0 \quad \text{as } h \rightarrow 0, \tag{5,2}$$

whatever the domain  $g_* \subset \bar{g}_* \subset g$ ;

$$3) \quad \iint_g \left( \frac{\partial^r \varphi}{\partial x^s \partial y^{r-s}} \right)^2 dg \leq M. \tag{5,3}$$

§ 6. **Lemma 2.** The formula holds

$$\iint_{g_\sigma} \left\{ \sum_{i=0}^l \frac{\partial^l v}{\partial x^{l-i} \partial y^i} \Delta_{\gamma,\beta-\gamma} \left( a_i \frac{\partial^l u}{\partial x^{l-i} \partial y^i} \right) + \sum_{i,p=0}^l \left[ \frac{\partial^l v}{\partial x^{l-p} \partial y^p} \Delta_{\gamma,\beta-\gamma} \left( b_{ip} \frac{\partial^l u}{\partial x^{l-i} \partial y^i} \right) + \frac{\partial^l v}{\partial x^{l-i} \partial y^i} \Delta_{\gamma,\beta-\gamma} \left( b_{ip} \frac{\partial^l u}{\partial x^{l-p} \partial y^p} \right) \right] \right\} \tag{6,1}$$

where  $v = \eta z$ ;  $z$  is an arbitrary function having a summable  $l$ -th derivative; the function  $\eta$  is defined by relation (3,1);  $\beta = 1, 2, \dots, q + 1$ ;  $\gamma = 0, 1, \dots, \beta$ .

**Lemma 3.** The formula holds

$$\begin{aligned} & \Delta_{\gamma,\beta-\gamma}[f(x,y)\varphi(x,y)] = \\ & = \sum_{s=0}^{\gamma} C_{\gamma}^s \sum_{k=0}^{\beta-\gamma} C_{\beta-\gamma}^k \Delta_{\gamma-s,\beta-\gamma-k} f(x,y) \Delta_{s,k} \varphi[x + (\gamma-s)h, y + (\beta-\gamma-k)h]. \end{aligned} \tag{6,2}$$

§ 7. We shall assume that the norm

$$\|f\|_{W_{2,-q}^{(p)}(g^*)}^2 = \begin{cases} \int_{g^*} \sum [D_p f]^2 t^{2q} dg^*, & p \geq l, \\ \sum_{i=1}^{l-1} \int_{g^*} \sum [D_i f]^2 t^{2i} dg^* + \int_g f^2 dg, & p = l - 1, \end{cases}$$

$g^* \subset g$ ; the sums are taken over all partial derivatives  $D_p f, D_i f$ . We note that the inequality obtained in work (4),

$$\|u\|_{W_{2,-l}^{(l)}(g)} \leq C_1(\delta) \|u\|_{W_{2,-(l-1)}^{(l-1)}(\Pi_{0,\delta})}, \quad (7,1)$$

is valid not only for constant, but also for variable  $a_i, b_{ip}$ .

The proof of the main inequality

$$\|u\|_{W_{2,-[l+r+\alpha(r+1)]}^{(l+r)}(g)} \leq C_r(\delta) \|u\|_{W_{2,-(l-1)}^{(l-1)}(\Pi_{0,\delta})} \quad (7,2)$$

(where  $C_r(\delta) > 0$  is a constant,  $r = 0, 1, \dots, q+1$ ) is carried out by induction. For  $r = 0$ , (7,2) is valid, since it coincides with (7,1). Suppose (7,2) is true for  $r = 0, 1, \dots, q$ . We shall show that (7,2) is true for  $r = q+1$ . To this end we put in the lemmas of § 6  $\beta = q+1$ ,  $z = \Delta_{\gamma, q+1-\gamma} u / h^{q+1}$ , divide equality (6,1) by  $h^{q+1}$  and substitute expression (6,2) into (6,1). We obtain  $(q+2)$  expressions equal to zero (which correspond to the values  $\gamma = 0, 1, \dots, q+1$ ). Estimating the terms containing under the integral sign  $[\Delta_{\gamma, q+1-\gamma} u^{(l)}]^2$  through the remaining ones and taking account of inequalities (2,1), (2,3), we obtain (for arbitrary  $m \geq 0$ )

$$\begin{aligned} & \iint_{g_\sigma} \sum_{i=0}^l \left( \frac{\partial^l \Delta_{\gamma, q+1-\gamma} u}{\partial x^{l-i} \partial y^i} \right)^2 \frac{(t-H)^2 (q+l+m+1+\alpha)}{h^2 (q+1)} dx dy \leq C(\delta) \times \\ & \times \max_{(s)} \left[ \iint_{g_\sigma} \sum_{i=0}^l \left( \frac{\partial^l \Delta_{\gamma, q+1-\gamma} u}{\partial x^{l-i} \partial y^i} \right)^2 \frac{|\eta^{(s)}|^\nu}{h^2 (q-1)} dx dy \right]^{1/2} + \\ & + \sum_{r=0}^q \left[ \iint_{g_0} \sum_{i=0}^l \left( \frac{\partial^l \Delta_{m, r-\mu}}{\partial x^{l-i} \partial y^i} \right)^2 \frac{|\eta^{(s)}|^{\nu_r}}{h^{2r}} dx dy \right]^{1/2} \times \\ & \times \max_{(i,p,\gamma,s)} \left[ \left( \iint_{g_0} \left( \frac{\Delta_{\gamma, q+1-r-\gamma} a_i}{h^{q+1-r}} \right)^2 |\eta^{(s)}|^{\nu''} dx dy \right)^{1/2}, \right. \\ & \left. \left( \iint_{g_0} \left( \frac{\Delta_{\gamma, q+1-r-\gamma} b_{ip}}{h^{q+1-r}} \right)^2 |\eta^{(s)}|^{\nu_r''} dx dy \right)^{1/2} \right] \times \end{aligned}$$

$$\times \left[ \iint_{g_0} \sum_{s=1}^l \sum_{i=0}^{l-s} \left( \frac{\partial^{l-s} \Delta_{\gamma, q+1-\mu}}{\partial x^{l-i-s} \partial y^i} \right)^2 \frac{1}{h^{2(q+1)}} |\eta^{(s)}|^\mu dx dy \right]^{1/2}, \quad (7.3)$$

where  $\gamma = 0, 1, \dots, r$ ;  $m = 0, 1, \dots, r$ ;  $C(\delta) > 0$  is a constant depending on  $\delta, q$ ;

$$|\eta^{(s)}| = \max_{(n)} \left| \frac{\partial^s \eta}{\partial x^n \partial y^{s-n}} \right|, \quad n = 0, 1, \dots, s.$$

We choose the positive number  $m$  entering the auxiliary function  $\lambda_m(t)$ . Put

$$m = \alpha(q+1), \quad \mu = \frac{2[l+q+1-s+\alpha(q+1)]}{2[l+(\alpha+1)(q+1)]-s},$$

$$\nu = \frac{2[l+q+1+\alpha(q+1)]}{2[l+(\alpha+1)(q+1)]-s}, \quad \nu'_r = \frac{2[l+r+\alpha(r+1)]}{2[l+(\alpha+1)(q+1)]-s}, \quad (7.4)$$

$$\nu''_r = \frac{2[q+1-r+\alpha(q-r)]}{2[l+(\alpha+1)(q+1)]-s}.$$

Then the necessary relations  $\nu + \mu = 2$  and  $\nu''_r + \nu'_r = \nu$  will be satisfied, and, by virtue of inequalities (2.1), (2.2), (2.3), (7.3), (7.4), letting  $h$ , and then also  $H$ , tend to zero (which is legitimate by Lemma 1), we obtain the inequality

$$\|u\|_{W_{2, -[l+q+1+\alpha(q+2)]}^{(l+q+1)}}^2 \leq$$

$$\leq C(\delta) \left\{ \|u\|_{W_{2, -[l+q+1+\alpha(q+2)]}^{(l+q+1)}}(\Pi_{0, \delta}) \sum_{s=1}^l \|u\|_{W_{2, -[l+q+1-s+\alpha(q+1)]}^{(l+q+1-s)}}(\Pi_{0, \delta}) + \right.$$

$$+ \sum_{s=1}^l \|u\|_{W_{2, -[l+q-s+1+\alpha(q+1)]}^{(l+q-s)}}(\Pi_{0, \delta}) \sum_{r=0}^q \|u\|_{W_{2, -[l+r+\alpha(r+1)]}^{(l+r)}}(\Pi_{0, \delta}) \times$$

$$\left. \times \max_{(i,p)} \left[ \|a_i\|_{W_{2, -[q+1-r+\alpha(q-r)]}^{(q+1-r)}}(\Pi_{0, \delta}) \|b_{ip}\|_{W_{2, -[q+1-r+\alpha(q-r)]}^{(q+1-r)}}(\Pi_{0, \delta}) \right] \right\}, \quad (7.5)$$

where  $C(\delta) > 0$  is a constant depending on  $\delta$ . The obtained inequality coincides with (7.2), as was required to prove.

In the case when at least one coefficient  $a_i, b_{ip}$  does not vanish near the boundary,  $\alpha = 0$  and formula (7.2) takes the form

$$\|u\|_{W_{2, -(l+r)}^{(l+r)}(g)} \leq C_r(\delta) \|u\|_{W_{2, -(l-1)}^{(l-1)}(\Pi_{0, \delta})}. \quad (7.6)$$

Inequalities (7.2) and (7.6) are sharp in the sense of the order  $l$ , as is shown by an example constructed in the work <sup>(4)</sup>.

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*Note: Figure translations are in progress. See original paper for figures.*

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