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Abstract

Full Text

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A BOUNDARY-VALUE PROBLEM WITH A COMPLEX WEIGHT FUNCTION

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In the present note we investigate the completeness of the system of eigen- and associated elements of certain classes of non-self-adjoint differential operators.

1. Consider in the Hilbert space $L^2(-\infty, +\infty)$ the differential equation

$$-y'' + q(x)y = \lambda p(x)y, \tag{1}$$

where $p(x) = q_1(x) + iq_2(x)$ is a complex function of the real parameter x, and $q(x), q_1(x), q_2(x)$ are real functions summable on every finite interval of the real axis.

Let us write equation (1) in the form

$$Ay = \lambda p(x)y, \tag{2}$$

where

$$A = -\frac{d^2}{dx^2} + q(x).$$

For $q(x) \geq 1$, A^{-1} is a self-adjoint positive operator (see ([3]), p. 75).

Put $y = A^{-1/2}z$, after which equation (2) is written as

$$AA^{-1/2}z = \lambda(q_1(x)A^{-1/2}z + iq_2(x)A^{-1/2}z). \tag{3}$$

Acting on equation (3) with the operator $A^{-1/2}$ and taking into account that A and $A^{-1/2}$ commute, we obtain:

$$z = \lambda(A^{-1/2}q_1(x)A^{-1/2}z + iA^{-1/2}q_2(x)A^{-1/2}z),$$

or

$$z = \lambda Lz,$$

where we have put

$$L = L_R + iL_I, \qquad L_R = A^{-1/2} q_1(x) A^{-1/2}, \qquad L_I = A^{-1/2} q_2(x) A^{-1/2}.$$

It is easy to verify that if $q_j(x) \geq 0$ $(j = 1, 2)$, then the real and imaginary components of the operator L are positive definite. We note that, for

$$\lim_{|x|\to\infty} \frac{q(x)}{|x|^\alpha} \geq C > 0 \quad \text{and} \quad \alpha \geq \frac{2}{3},$$

A^{-1} is an operator of Hilbert-Schmidt type (see ([6]), p. 6).

Assume that $q_1(x)$ is a bounded function. Then the operator $q_1(x)A^{-1}$, being the product of the bounded $q_1(x)$ and the Hilbert-Schmidt operator A^{-1}, is an operator of Hilbert-Schmidt type. Using the inequality (see ([4]), p. 12)

$$N^2(A^{-1/2} q_1(x) A^{-1/2}) \leq \frac{1}{2} N^2(A^{-1} q_1(x)) + \frac{1}{2} N^2(q_1(x) A^{-1}),$$

where $N(A)$ is the absolute norm of the operator A, we conclude that the operator $A^{-1/2} q_1(x) A^{-1/2}$ is of Hilbert-Schmidt type. By an analogous argument we obtain that $A^{-1/2} q_2(x) A^{-1/2}$ is a Hilbert-Schmidt operator under the condition that $q_2(x)$ is bounded.

Consequently, L is an operator of Hilbert-Schmidt type.

Applying Theorem 2 of V. B. Lidskii ([2]), we arrive at the following result.

Theorem 1. *If $q(x) \geq 1$ and a) $q_1(x)$, $q_2(x)$ are nonnegative and bounded functions, b)*

$$\lim_{|x|\to\infty} \frac{q(x)}{|x|^\alpha} \geq C > 0$$

for $\alpha \geq \frac{2}{3}$, then the eigenvectors and associated vectors of the operator L, corresponding to nonzero points of the spectrum, form a system complete in the range of values of the operator L.

Remark. If $q_1(x) > 0$ and $q_2(x) > 0$, then the eigenvectors and associated vectors of the operator L, corresponding to nonzero points of the spectrum, form a system complete in $L^2(-\infty, +\infty)$.

2. We now consider an equation of the form

$$-y'' + q(x)y = \lambda p_1(x)y + \frac{p_2(x)}{\lambda}y + p_3(x)y, \tag{4}$$

where $p_j(x) = q_j(x) + iq_j'(x)$, $j = 1, 2, 3$, are complex functions of the real parameter x, and $q(x)$, $q^j(x)$, $q_j'(x)$ are real functions summable on every finite interval of the real axis.

We shall show that, under certain restrictions on $q(x)$ and $p_j(x)$, the system of eigenvectors and associated vectors of equation (4) is complete in $L^2(-\infty, +\infty)$.

In what follows we shall use a theorem of D. E. Allakhverdiev ([5]).

Let A, B, C be completely continuous operators in a Hilbert space $\mathfrak{H}$; suppose $(E-A)^{-1}$ exists and is bounded; A and C have finite order, respectively ρ_1 and ρ_2. Suppose, moreover, that the resolvents of the operators A and C are bounded respectively on rays $\alpha_1, \alpha_2, \dots, \alpha_n$ and $\beta_1, \beta_2, \dots, \beta_n$, such that the angle between neighboring rays is respectively less than π/ρ_1 and π/ρ_2, and the operators A^* and C^* are invertible. Then the system of eigenvectors and associated vectors of the equation

$$y = \lambda Ay + \frac{B}{\lambda}y + Cy$$

is 2-fold complete in the Hilbert space $\mathfrak{H}$.

Recall that the order of an operator A is the lower bound of the numbers α for which

$$\sum_{n=0}^{\infty} \varepsilon_n^{\alpha}(A) < +\infty,$$

where

$$\varepsilon_0(A) = \|A\|; \qquad \varepsilon_n(A) = \inf_{A^{(n)}} \|A - A^{(n)}\| \quad \text{for } n \geq 1;$$

$A^{(n)}$ runs through the set of all n-dimensional operators acting in $\mathfrak{H}$ (see ([4]); there D. E. Allakhverdiev also proves that $\varepsilon_n(A) = 1/\sqrt{\lambda_{n+1}}$, where $\{\lambda_k\}$ are the eigenvalues of the operator AA^*). Thus, the order of the operator A (in the sense of ([4])) coincides with the order of the operator $(AA^*)^{1/2}$ in the sense of M. V. Keldysh.

We transform equation (4) to the form:

$$z = \lambda A^{-1/2}p_1(x)A^{-1/2}z + \frac{A^{-1/2}p_2(x)A^{-1/2}}{\lambda}z + A^{-1/2}p_3(x)A^{-1/2}z.$$

For $\alpha > \dfrac{2}{3}$, A^{-1} is an operator of finite order less than 2. As we noted, the order of the operator $A^{-1/2}p_j(x)A^{-1/2}$ coincides with the order of the operator

$$\left(A^{-1/2}p_j(x)A^{-1/2}A^{-1/2}p_j(x)A^{-1/2}\right)^{1/2},$$

and the order of the latter, according to the result—

there, by M. V. Keldysh, does not exceed the order of the operator A^{-1} (see (4)). Thus, $A^{-1/2}p_j(x)A^{-1/2}$ is an operator of finite order less than 2.

Suppose that $q_j(x)$ and $q_j'(x)$ have a definite sign. Let, for example, $q_j(x) \geq 0$ and $q_j'(x) \leq 0$. Then the values of the quadratic form $(A^{-1/2}p_j(x)A^{-1/2}f, f)$ lie in the second quadrant, and for $\rho_1 < 2$ and $\rho_2 < 2$, on the basis of a lemma of V. B. Lidskii (see (7)), on rays issuing from the origin and not falling into the second quadrant, the resolvent $(E - A^{-1/2}p_1(x)A^{-1/2})^{-1}$ and the resolvent $(E - A^{-1/2}p_3(x)A^{-1/2})^{-1}$ exist and are bounded.

Thus, we have proved the following theorem.

Theorem 2. If $q(x) \geq 1$ and

a) $\lim\limits_{|x|\to\infty} \dfrac{q(x)}{|x|^\alpha} \geq C > 0$ for $\alpha > \frac{2}{3}$;

b) $\overline{p_j(x)} \neq 0$ almost everywhere and $q_j(x)$, $q_j'(x)$ are bounded functions of definite sign;

c) the equation $Az = p_1(x)z$ has no solutions other than the zero solution,

then the system of eigen- and associated elements of equation (4) is complete in $L^2(-\infty, +\infty)$.

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Note: Figure translations are in progress. See original paper for figures.