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Abstract

Full Text

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ON THE APPROXIMATION OF CONTINUOUS FUNCTIONS BY HARMONIC ONES

(Presented by Academician M. V. Keldysh on 17 IX 1963)

1. Let E be an arbitrary compact set in three-dimensional Euclidean space C^3; $C(E)$ is the Banach space of all real-valued functions $f(P)$, $P \in E$, continuous on E, with norm

$$\|f\| = \max_{P\in E} |f(P)|.$$

Denote by $H(E)$ the closure in the norm of the space $C(E)$ of the set of functions harmonic on the compact set E (i.e., admitting a harmonic continuation into some neighborhood, depending on the function, of the compact set E). The elementary harmonic functions with respect to a point A are the functions of the form

$$H_n^A(P) = \frac{H_n(P)}{r^{2n+1}}, \qquad r = r(P, A) = \overline{PA},$$

where $H_n(P)$ is a homogeneous harmonic polynomial of degree n, $H_n^\infty(P) = H_n(P)$. $H(E)$ coincides with the closed linear span of the elementary harmonic functions $\{H_n^A(P)\}$, provided that the set $\{A\}$ has a nonempty intersection with each of the complementary domains of the compact set E; in particular, if the complement of E consists of one domain, then $H(E)$ consists of those and only those functions which admit uniform approximation by harmonic polynomials.

We shall denote by $K(P, \delta)$ the open ball of radius δ with center at the point P, and by $\gamma(P, \delta)$ the capacity (with respect to the Newtonian potential; see, for example, (1)) of that part of the complement of E which belongs to the ball $K(P, \delta)$.

2. The question of the possibility of uniformly approximating continuous functions by harmonic polynomials was investigated in the works of M. V. Keldysh and M. A. Lavrent' ev. M. V. Keldysh (1) solved the general problem of the conditions on a closed domain $\overline{D}$ under which any function continuous in $\overline{D}$ and harmonic on the set of interior points of D admits uniform approximation by harmonic polynomials; in (1) the corresponding result is formulated in terms of the stability in $\overline{D}$ of every solvable

Dirichlet problem. In the works of Brelot (2) and Deny (3) Keldysh' s theorem was extended to the case of approximation by harmonic functions on an arbitrary compact set E; in particular, the answer to the question under what conditions on a compact set E every continuous function on E can be uniformly approximated by harmonic functions is given by the following theorem (2,3), cf. also the work of N. S. Landkof (4).

Theorem A. In order that $C(E) = H(E)$, it is necessary and sufficient that at each point $P \in E$ the condition

$$\int_0 \frac{\gamma(P,\delta)}{\delta^2}\, d\delta = \infty. \tag{1}$$

be satisfied.

The divergence of the integral (1) at a given point $P \in E$ is the necessary and sufficient condition (in the Kellogg-Wasilewski form) for the regularity of the point P; this condition was first obtained by M. V. Keldysh (1) in a somewhat different form (the divergence of the Wiener-Keldysh series).

In the present paper we shall give some results that adjoin Theorem A; all these results are obtained on the basis of the analogy between the problem of uniform approximation by harmonic functions in space and the problem of uniform approximation by analytic (or rational) functions in the plane of a complex variable (5-7) (see Remark 1 below).

3. The following theorem shows that capacity has the property of "instability" :

Theorem 1. *If for almost all (with respect to measure in C^3) points $P \in E$ the condition*

$$\overline{\lim_{\delta\to 0}} \frac{\gamma(P,\delta)}{\delta^3} = \infty, \tag{2}$$

is satisfied, then for all $P \in C^3$ and all $\delta > 0$

$$\gamma(P,\delta) \geq A_0\delta, \tag{3}$$

where $A_0 > 0$ is an absolute constant.

Theorem 1 is an analogue of a theorem of A. G. Vitushkin (7) on the property of "instability" of the analytic capacity (Ahlfors measure) of plane sets.

If at some point P, for every $\delta > 0$, condition (3) is satisfied, then at this point condition (1) is also satisfied; in turn, (1) implies (2). Consequently, the divergence of integral (1), as a necessary and sufficient condition for the possibility of approximation on nowhere dense sets E (but not as a local condition for a point of stability), is accidental in character. It is natural to formulate the theorem on the possibility of approximation in the following way:

Theorem B. *Each of the following conditions is necessary and sufficient in order that $C(E) = H(E)$:*

a)
$$\overline{\lim_{\delta\to 0}} \frac{\gamma(P,\delta)}{\delta^3} = \infty$$
for almost all $P \in E$;

b) *for all $P \in C^3$ and $\delta > 0$,*
$$\gamma(P,\delta) \geq A_0\delta,$$
where $A_0 > 0$ is an absolute constant.

4. **Lemma 1.** *Let e be a closed set of positive capacity $c(e)$, contained in the ball $K(Q,\delta)$; let $W_e(P)$ be the potential of the set e (see (^{1})). Then*
$$W_e(P) = \sum_{n=1}^{\infty} \frac{B_n(\theta,\varphi)}{r^n}, \qquad r = r(P,Q) > \delta,$$
where $B_1 = c(e)$ and
$$|B_n(\theta,\varphi)| \leq A_1 n^2 \delta^{n-1} c(e); \tag{4}$$
A_1 is an absolute constant, and (r,θ,φ) are spherical coordinates with center at the point Q.

Estimate (4) is valid for the coefficients of the expansion of an arbitrary function $U(P)$ satisfying the conditions: 1) $U(P)$ is harmonic in the complement of e; 2) $U(P) \to 0$ as $r \to \infty$ with rate $\frac{c}{r}$; 3) $|U(P)| \leq 1$, $P \in e$.

Lemma 2. *Let a closed set e of positive capacity $c(e)$ be contained in $K(Q,\delta)$. Then there exists a function $U_Q(P)$, harmonic outside the set e, such that:*

1)
$$\left|U_Q(P) - \frac{1}{r}\right| < A_2\frac{\delta}{r^2}, \qquad r = r(P,Q) > 2\delta;$$

2)
$$|U_Q(P)| \leq \frac{1}{c(e)}, \qquad P \in e.$$

Lemma 3. *Let E be a compact set, $\delta > 0$,*
$$\gamma(\delta) = \inf_{Q\in E} \gamma(Q,\delta).$$
For any point $Q \in E$ there exists a function $U_Q(P)$ such that:

1) $U_Q(P)$ is harmonic on E;

2) $\left|U_Q(P) - \frac{1}{r}\right| < A_2 \frac{\delta}{r^2}, \quad r = r(P,Q) > 2\delta, \quad P \in E;$

3) $|U_Q(P)| < \frac{2}{\gamma(\delta)}, \quad P \in E;$

4) there exists a finite number of points $Q_1, \dots, Q_N$ belonging to E and such that for any point $Q \in E$, $U_Q(P) = U_{Q_k}(P)$, where Q_k is one of the points Q_i, $i = 1, \dots, N$.

The lemmas stated above make it possible to transfer to the case of space the well-known method of S. N. Mergelyan [5] for proving approximation theorems. In particular, based on these lemmas, one can give a direct (not relying on theorems A and 1) constructive proof of the following theorem:

Theorem 2. If

$$\lim_{\delta \to 0} \frac{\gamma(\delta)}{\delta^3} = \infty, \qquad \gamma(\delta) = \inf_{P \in E} \gamma(P, \delta), \tag{5}$$

then $C(E) = H(E)$.

It is enough to prove the possibility of uniform approximation on E of an arbitrary polynomial $\Phi(x, y, z) = \Phi(P)$. Approximating the compact set E by sufficiently "good" open sets D_n and applying Green' s formula, we reduce the question of approximation of $\Phi(x, y, z)$ to the corresponding question for the function

$$I(P) = \iiint_E \frac{\Delta\Phi(Q)}{r(P,Q)}\, dv_Q,$$

where Δ is the Laplace operator. Let $U_Q(P)$ be the function whose existence is asserted in Lemma 3,

$$U(P) = \iiint_E U_Q(P) \cdot \Delta\Phi(Q)\, dv_Q.$$

$U(P)$ is a function harmonic on E, and

$$\|I(P) - U(P)\| \le A_3 M \left(\delta + \frac{\delta^3}{\gamma(\delta)}\right),$$

where $M = \|\Delta\Phi(P)\|$, and A_3 is an absolute constant; hence, applying (5), we obtain the assertion of the theorem.

5. The condition (5) of Theorem 2 (and, a fortiori, condition a) of Theorem B) cannot be substantially weakened:

Whatever function $\mu(\delta) \to 0$ as $\delta \to 0$ may be, there exists a compact set E such that $C(E) \neq H(E)$ and

$$\lim_{\delta\to 0} \frac{\gamma(\delta)}{\mu(\delta)\delta^3} = \infty.$$

In particular, Theorem 2 ceases to be true if condition (5) is replaced by the condition

$$\lim_{\delta\to 0} \frac{\gamma(\delta)}{\delta^{3+\alpha}} = \infty, \qquad \alpha > 0.$$

6. Completely analogous results are also valid for the case of approximation of continuous functions in m-dimensional Euclidean space C^m, $m > 3$; preserving the notation adopted above, we formulate the analogue of Theorem B:

Theorem B′. Each of the following conditions is necessary and sufficient in order that $C(E) = H(E)$:

a) $\lim\limits_{\delta\to 0} \dfrac{\gamma(P,\delta)}{\delta^m} = \infty$ for almost all (with respect to the measure of the space C^m) $P \in E$;

b) for all $P \in C^m$ and $\delta > 0$, $\gamma(P,\delta) \geq A_0\delta$; $A_0 > 0$ is an absolute constant.

Remark 1. The problem of the possibility of uniform approximation of an arbitrary continuous function on a compact set E in the plane of the complex variable by functions analytic on E (or by rational functions with poles outside E) is closely connected with the problem of the possibility of uniform approximation on E of an arbitrary real continuous function by real parts of functions analytic on E (or rational functions); the conditions on E that are necessary and sufficient for the possibility of approximation in these cases simply coincide (see ([8])). The class of all functions harmonic on a plane compact set E is substantially wider than the class of those functions harmonic on E which are real parts of functions analytic on E, since the conjugate function may be multivalued; therefore the problems of harmonic and analytic approximation in the plane are essentially different problems. In this connection it is interesting to note that, although the problem of approximation by harmonic functions in the space C^m, $m \geq 3$, coincides in its formulation with the problem of harmonic approximation in the plane, in the form of its solution given in the present paper it is rather analogous to the problem of analytic approximation in the plane; to see this it suffices to compare the formulations of Theorems B and B′ with the formulation of A. G. Vitushkin’ s theorem ([7]).

Remark 2. Let E be a compact set in the plane of the complex variable. Denote by $C_0(E)$ the totality of all functions continuous on E and analytic on the set of interior points of E; if

$$\|f\| = \max_{z \in E} |f(z)|,$$

then $C_0(E)$ is a Banach algebra (^{8}). Let $A(E)$ be the subalgebra of the algebra $C_0(E)$ consisting of all functions that admit uniform approximation by functions analytic on E (or rational functions with poles outside E). Up to now no conditions have been found that are necessary and sufficient in order that $C_0(E) = A(E)$.

Let M_0 and M be the minimal boundaries (^{8}) of the algebras $C_0(E)$ and $A(E)$, respectively; obviously, $M \subset M_0 \subset \partial E$, where ∂E is the topological boundary of E; M_0 is the analogue of the set of regular points, M of the stable points of the Dirichlet problem (^{1}). By analogy with the general theorem on approximation by harmonic functions in space (^{1,2}), it is natural to state the hypothesis:

In order that the equality $C_0(E) = A(E)$ hold, it is necessary and sufficient that $M_0 = M$. In particular, if $M = \partial E$, then $C_0(E) = A(E)$.

The following theorem is true (^{9}):

If

$$\lim_{r \to 0} \frac{d(\zeta, r)}{r} > 0,$$

where $d(\zeta, r)$ is the upper bound of the diameters of the connected components of the intersection of the complement of E with the disk $|z - \zeta| < r$, then $\zeta \in M$. In particular, if the point ζ is a boundary point of some component of the complement of E, then $\zeta \in M$.

Already this simple condition for a point ζ to belong to the minimal boundary M of the algebra $A(E)$ makes it possible to obtain, as direct consequences of the hypothesis formulated above, all known geometric sufficient conditions (see (^{5})) for the coincidence of $A(E)$ with $C_0(E)$, in particular, Mergelyan' s general theorem on uniform approximation by polynomials. We also note one consequence which remains unproved: if every boundary point of the compact set E is a boundary point of some component of the complement of E, then $C_0(E) = A(E)$.

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Note: Figure translations are in progress. See original paper for figures.

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