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Abstract

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ON THE POSSIBLE MAGNITUDE OF THE LOWER ORDER OF AN ENTIRE FUNCTION WITH A FINITE DEFICIENT VALUE

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Let $f(z)$ be an entire function of order ρ and lower order λ. We shall use the standard notation of Nevanlinna theory (1). It is known that if $\rho < \infty$ and $f(z)$ has a finite Picard value, then $\lambda = \rho$ and ρ is an integer. If $f(z)$ has a finite deficient value in the sense of Nevanlinna, then it may happen that $\lambda < \rho$, and the problem arises: to determine what values the lower order can assume for entire functions of fixed finite order ρ possessing a finite deficient value. This problem was included in the list of problems compiled by the participants of the Colloquium on the classical theory of functions, held in 1961 at Cornell University (2,3). Earlier Edrei and Fuchs (4) had put forward the hypothesis that λ and ρ are related by the inequality $\lambda \geq \rho/2$. This hypothesis is cited in (2,3) as the probable solution of the problem. From a theorem of I. V. Ostrovskii (5) it follows that for $\lambda < 1/2$ an entire function cannot have finite deficient values (for $\lambda < 1/2$ this already follows from a result of Kjellberg (6)). In the paper (7) we showed that there exists an entire function with defect $\delta(0) > 0$ and with arbitrary prescribed λ and ρ such that $1 \leq \lambda \leq \rho < \infty$. Thus the Edrei-Fuchs hypothesis was disproved, but the problem was not completely solved. Here we construct an example of an entire function with $\delta(0) > 0$ and arbitrary λ and ρ such that $1/2 < \lambda \leq \rho < \infty$. Together with the result of I. V. Ostrovskii mentioned above, this gives a complete answer to the question posed in (2,3)*.

Let numbers λ and ρ be given such that $1/2 < \lambda \leq \rho < \infty$. Choose an integer odd number $n > \rho$ and a number α such that $\alpha \leq \lambda$ and $1/2 < \alpha < 1$. Let

$$g(\zeta) = \prod_{k=1}^{\infty}(1 - \zeta k^{-\beta}), \qquad \text{where } \beta = 1/\alpha.$$

It is known (8) that there exists $s_0(\eta)$ such that for $s > s_0$ and for all θ, $0 \leq \theta \leq 2\pi$,

$$\ln|g(se^{i\theta})| < s^{\alpha}\{\eta + \pi \cos\alpha(\theta - \pi)\operatorname{cosec}\pi\alpha\},$$

where $0 < 2\eta < \pi|\cos\alpha\pi|\operatorname{cosec}\pi\alpha$. The function $w = g(\zeta)$ maps the finite ζ-plane onto a certain Riemann surface F_1, which has the following structure (9). Let ξ_k be the sequence of zeros of $g'(\zeta)$, $k^\beta < \xi_k < (k+1)^\beta$, $k = 1, 2, \dots$, $a_k = g(\xi_k)$, $\operatorname{sgn} a_k = (-1)^k$. Denote by B_0 the copy of the w-plane with a cut along $(-\infty, a_1)$, and by B_j the copy of the w-plane from which the points of the real axis have been removed except for the interval with endpoints at a_j and a_{j+1}, $j = 1, 2, \dots$. By gluing the sheets B_{j-1} and B_j along the cut from a_j to $(-1)^j\infty$, $j = 1, 2, \dots$, we obtain the surface F_1. For $|a_k|$ in (9) the estimate

$$|a_k| < (\pi k)^{-1}, \qquad k = 1, 2, \dots$$

is established**. Denote by K_1 the simply connected piece of the Riemann surface, $K_1 \subset F_1$, which consists of the intersections

$$B_k \cap \{|w| < 2\}, \qquad k = 0, \dots, n-2,$$

and

$$B_{n-1} \cap \{|w| < 2,\ \operatorname{Re} w > 0\}.$$

The boundary of K_1 consists from the spiral σ over $|w| = 2$ and the segment $l = (-2i, 2i)$. Let $F_0 = F_1 - K_1$. Denote by K_2 the part of the Riemann surface of the function $(w-1)^{1/n}$ lying over $|w| < 2$, where on one of the sheets the semicircle $\{|w| < 2,\ \operatorname{Re} w \le 0\}$ is absent. The boundary of K_2 also consists of the spiral σ and the segment l. Gluing the surfaces F_0 and K_2 along σ and l, we obtain a simply connected Riemann surface F_2, which, evidently, is of parabolic type. The surface F_2 has an algebraic branch point of order $n-1$ over $w = 1$ and simple branch points over the points $w = a_n, a_{n+1}, \dots$. Let L be an infinite polygonal line on F_2 with consecutive vertices at the indicated branch points. Make a cut along L, and also along the ray $\{\operatorname{Im} w = 0,\ \operatorname{Re} w > 1\}$ on that sheet of the surface F_2 whose intersection with F_0 coincides with $B_0 \cap F_0$. Then F_2 splits into two symmetric parts. Let $h(z)$ be an entire function mapping the finite z-plane ($z = x + iy = re^{i\varphi}$) onto F_2, with the positive real half-axis passing into L,

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* We note that it is not known to us whether there exists an entire function with a finite deficient value and with $\lambda < \rho = \infty$.

** Using a result of Wiman (10), it is easy to establish that

$$|a_k| \sim (2\pi)^{-\beta/2} k^{-1/2} \times \exp(\pi k^\alpha \operatorname{ctg}\pi\alpha)$$

as $k \to \infty$.

$h(0) = 1$. We indicate some properties of $h(z)$. The function $h(z)$ is real entire: for $x > 0$ we have $|h(x)| < 1$, since $h(x) \in L$;

$$h(z) = 1 + c_n z^n + c_{n+1} z^{n+1} + \cdots, \tag{1}$$

where $c_n < 0$ (the latter follows from the fact that $h(x)$ is a decreasing function in a sufficiently small neighborhood of $x = 0$). It is easy to find the asymptotics of $h(z)$. Denote by G_1 and G_2 the images of F_0 under the mappings respectively by the functions $\zeta = g^{-1}(w)$ and $z = h^{-1}(w)$. Evidently, G_1 and G_2 are doubly connected domains with an infinitely distant point as one of the boundary components, symmetric with respect to the real axis; moreover, $\zeta = \zeta(z) = g^{-1}[h(z)]$ effects a conformal one-sheeted mapping of G_2 onto G_1, and in a neighborhood of $z = \infty$ we have an expansion of the form $\zeta(z) = dz + d_0 + \cdots$, where $d > 0$. Without loss of generality, the function $h(z)$ may be considered normalized so that $d = 1$. Since $h(z) = g[\zeta(z)]$, for all $r > R_0 > s_0$ and all φ, $0 \le \varphi \le 2\pi$, the inequality

$$\ln|h(re^{i\varphi})| < r^\alpha\{2\eta + \pi \cos\alpha(\varphi - \pi) \operatorname{cosec} \pi\alpha\} \tag{2}$$

holds.

We shall denote by C with subscripts positive constants which may depend only on α and n. Taking (1) and (2) into account, it is easy to see that $\ln M(r, h) < C_1\Psi(r)$, $0 < r < \infty$, where $\Psi(r) = r^{n+\alpha}(r^\alpha + r^n)^{-1}$. Using (1), we find that there exist such R_1, $0 < R_1 < R_0$, and φ_1, $0 < \varphi_1 < \pi/2n$, that for all $r \in [0, R_1]$ and $|\varphi| \le \varphi_1$,

$$\ln|h(re^{i\varphi})| \le -C_2 r^n.$$

From (2) there follows the existence of such $R_2 > R_0$ and φ_2, $0 < \varphi_2 < \pi(1 - \beta/2)$, that for all $r > R_2$ and $|\varphi| \le \varphi_2$ one has

$$\ln|h(re^{i\varphi})| \le -C_3 r^\alpha.$$

When $x \in [R_1, R_2]$, we have $\ln|h(x)| < 0$, and, by virtue of the uniform continuity of $|h(re^{i\varphi})|$, $r \le R_2$, there exists such φ_3, $0 < \varphi_3 < \pi$, that for all $r \in [R_1, R_2]$ and $|\varphi| \le \varphi_3$ the estimate

$$\ln|h(re^{i\varphi})| \le -C_4$$

holds. Combining these estimates, we obtain that for all r, $0 \le r < \infty$, and $|\varphi| \le \varphi_0 = \min(\varphi_1, \varphi_2, \varphi_3)$ the inequality $\ln|h(re^{i\varphi})| \le -C_5\Psi(r)$ is valid.

Let $\{r_k\}$ be a nondecreasing sequence of positive numbers tending to $+\infty$ such that $\sum r_k^{-n} < \infty$. Denote by $\nu(r)$ the number of points r_k on the segment $[0, r]$. By virtue of (1), the infinite product

$$f(z) = \prod_{k=1}^{\infty} h(z/r_k)$$

converges absolutely and represents an entire function. Taking into account the estimates for $h(z)$, we obtain that

$$\ln M(r, f) \leq C_1 \sum_{k=1}^{\infty} \Psi(r/r_k) = C_1 \Phi(r) \tag{3}$$

and for all $|\varphi| \leqslant \varphi_0$ one has $\ln |f(re^{i\varphi})| \leqslant -C_5 \Phi(r)$, whence it follows that

$$m(r, 0, f) \geqslant C_6 \Phi(r), \qquad C_6 = \frac{\psi_0}{\pi} C_5. \tag{4}$$

Comparing (3) and (4), we find

$$\delta(0, f) \geqslant \liminf_{r \to \infty} m(r, 0, f) / \ln M(r, f) \geqslant C_7 > 0,$$

where $C_7 = C_6 C_1^{-1}$. Thus, for any choice of the sequence r_k, the entire function $f(z)$ has at zero a defect not less than a certain positive number C_7.

We shall now show that the sequence r_k can be chosen so that the function $f(z)$ has prescribed in advance ρ and λ, $\alpha \leqslant \lambda \leqslant \rho < n$. Observe that, in view of (3) and (4), we have $C_6 \leqslant \ln M(r, f)/\Phi(r) \leqslant C_1$, and therefore the order and the lower order of $\ln M(r, f)$ and $\Phi(r)$ coincide. Writing $\Phi(r)$ in the form of a Stieltjes integral, integrating by parts and noting that $v(r) = 0$ for $0 \leqslant r < r_1$ and $v(r) = o(r^n)$ as $r \to \infty$, we obtain

$$\Phi(r) = \int_0^{\infty} v(t) \Psi'\left(\frac{r}{t}\right) r \frac{dt}{t^2} = \int_0^{\infty} v(r\tau) \Psi'\left(\frac{1}{\tau}\right) \frac{d\tau}{\tau^2}.$$

It is easy to verify that

$$C_8 < \Psi'(1/\tau) \tau^{\alpha - 1} (1 + \tau^{n - \alpha}) < C_9,$$

and therefore

$$C_9^{-1} \Phi(r) < \int_0^{\infty} \frac{v(r\tau)}{\tau^{1+\alpha}} \frac{d\tau}{1 + \tau^{n-\alpha}} < C_8^{-1} \Phi(r).$$

It is now clear that it suffices to choose the function $v(r)$ so that: 1) for all r the inequalities $r^\lambda - 1 \leqslant v(r) \leqslant r^\rho + 1$ hold; 2) on some sequence of intervals $(\gamma_k, \exp\exp\gamma_k)$, $\gamma_k \to \infty$, one has $v(r) \leqslant r^\lambda + 1$; 3) on a sequence of intervals $(\omega_k, \exp\exp\omega_k)$, $\omega_k \to \infty$, $v(r) \geqslant r^\rho - 1$ holds. Then the function $\Phi(r)$, and with it also $f(z)$, will have order ρ and lower order λ.

Remark. If one requires only that $1 \leqslant \lambda \leqslant \rho < \infty$, then, using the same idea, the desired example can be constructed much more simply.

For the function $h(z)$ we take

$$h(z) = e^{-z} \sum_{k=0}^{n-1} z^k/(k!), \qquad \alpha = 1.$$

Then the preceding arguments can be repeated, but the investigation of the properties of $h(z)$ is simplified, since

$$h'(z) = -e^{-z} z^{n-1}/(n-1)!.$$

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