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Abstract

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MATHEMATICS

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HOMEOMORPHISM OF A SYSTEM OF DIFFERENTIAL EQUATIONS IN NEIGHBORHOODS OF CLOSED TRAJECTORIES

(Presented by Academician A. N. Kolmogorov on 26 X 1963)

Consider the system of differential equations

$$\frac{dx}{dt} = P(x), \quad (1)$$

where $P(x)$ is an n -dimensional vector of class C^1 , defined in some domain of an n -dimensional vector space.

Definition 1. We shall call a system of differential equations in a domain D_1 homeomorphic to another system of differential equations in a domain D_2 if there exists a homeomorphism of the domain D_1 onto D_2 under which the points of one and the same trajectory of the first system are mapped onto points of one and the same trajectory of the second system.

D. M. Grobman ⁽¹⁾ proved the homeomorphism of two systems (1) in neighborhoods of singular points, provided only that the solutions have no zero characteristic exponents and that the number of negative characteristic exponents is the same for both systems. In the present note an analogous theorem is given on the homeomorphism of two systems of differential equations in neighborhoods of two closed trajectories.

Definition 2. By the **multipliers** of a closed trajectory of the system of differential equations (1) we shall mean the eigenvalues of the matrix of the linear part of the section mapping of a neighborhood of the closed trajectory into the same section, defined by the mapping of the point of intersection of some trajectory of the system (1) with the given section to the next point of intersection of the same trajectory with the same section.

Theorem. *If two closed trajectories of certain systems of the form (1) have the same number of multipliers, in absolute value less than one, among which*

there is the same number of negative ones, and the same number of negative multipliers, in absolute value greater than one, and have no multipliers whose absolute value is equal to one, then these systems are homeomorphic in certain neighborhoods of the given closed trajectories.

The proof is carried out as follows. Let the system (1) have a closed trajectory Γ . This closed trajectory has a neighborhood D in which one can introduce a local coordinate system (s, y) so that $y = 0$, $0 \leq s < 1$ gives all points of Γ (y is an $(n - 1)$ -dimensional vector), and all points on one and the same hyperplane transversal to Γ have one and the same coordinate s ; moreover, on passing to the coordinate system (s, y) from the system (1) one obtains a system whose right-hand sides belong to the class C^1 (2), i.e., the system

$$\frac{dy}{ds} = A(s)y + f(s, y), \quad (2)$$

where $A(s + 1) = A(s)$, $f(s + 1, y) = f(s, y)$, and as $|y| \rightarrow 0$, $|\frac{\partial f}{\partial y}| = o(1)$ ($|y|$ denotes the norm of the vector, and $|\partial f / \partial y|$ the norm of the $(n - 1) \times (n - 1)$ -matrix). If (s, y) are regarded as Cartesian coordinates, then the transforma-

the equation $x = \Psi(s, y)$ gives a mapping of a neighborhood U of the s -axis onto a neighborhood D of the closed trajectory Γ .

Choose R so small that $|y| < 2R$ is contained in U . Construct a function $g(s, y)$ such that $g(s, y) = f(s, y)$ in some sufficiently small neighborhood U_1 contained inside the neighborhood $|y| < R$; $g(s, y) = 0$ if $|y| \geq R$; $g(s + 1, y) = g(s, y)$, and $g(s, y)$ satisfies a Lipschitz condition for all y with the same constant as $f(s, y)$ in U_1 . Instead of system (2) we shall consider the system

$$\frac{dy}{ds} = A(s)y + g(s, y). \quad (3)$$

Along with system (3) we shall consider the system

$$\frac{dz}{ds} = A(s)z. \quad (4)$$

System (4), by a Lyapunov transformation (3), passes into the system

$$\frac{dv}{ds} = B_2 v, \quad (5)$$

where $K_2(s)$ is a nonsingular matrix, $K_2(s + 2) = K_2(s)$, B is a constant canonical matrix having only real or complex-conjugate eigenvalues and having no eigenvalues with zero real part. Let the diagonal elements of the matrix B_2 be arranged in order of increasing real parts. Denote by $V(s)$ the normal fundamental matrix of system (5), $V(0) = E_{n-1}$ ($(n - 1) \times (n - 1)$ is the identity

matrix). Let $V_1(s)$ be the matrix obtained from $V(s)$ if in it one replaces by zeros those columns which have positive characteristic exponents, $V_{-1}(s) = V_1(-s)$, $V_2(s) = V(s) - V_1(s)$, $V_{-2}(s) = V_2(-s) = V^{-1}(s) - V_{-1}(s)$, $Z(s) = K_2^{-1}(s)V(s)$, $Z_i(s) = K_2^{-1}(s)V_i(s)$, $Z_{-i}(s) = V_{-i}(s)K_2(s)$, $i = 1, 2$.

Denote by $y(s, y_0, s_0)$ the trajectory of system (3) passing, for $s = s_0$, through the point (s_0, y_0) .

Definition 3. We shall call a homeomorphism $(s, z) = \Phi(s, y)$, mapping some system of differential equations (3) in some neighborhood of the s -axis in the space (s, y) onto another system of differential equations of the same form in a neighborhood of the s -axis in the space (s, z) , **periodic** if from $(s_1, z_1) = \Phi(s_1, y_1)$ it follows that $(s_1 + m, z_1) = \Phi(s_1 + m, y_1)$, $m = 0, \pm 1, \pm 2, \dots$

Lemma 1. *The mapping $(s_1, z_1) = \Phi_0(s_1, y_1)$, defined by the formulas*

$$s_1 = s_1,$$

$$z_1 = y_1 - Z_1(s_1) \int_{-\infty}^0 Z_{-1}(\sigma + s_1)g(\sigma + s_1, y(\sigma + s_1, y_1, s_1)) d\sigma + Z_2(s_1) \int_0^{\infty} Z_{-2}(\sigma + s_1)g(\sigma + s_1, y(\sigma + s_1, y_1, s_1)) d\sigma, \quad (6)$$

gives a periodic homeomorphism of system (2) in a neighborhood U_1 onto system (4) in some neighborhood U_2 of the s -axis.

The proof of Lemma 1 is carried out analogously to the proof of the theorem in ⁽¹⁾. In addition, the periodicity of the homeomorphism (6) is proved. The latter follows from the diagonal-block form of the principal matrix C of the fundamental system $Z(s)$, the periodicity of $g(s, y)$, and the relation $y(s + m, y_1, s_1 + m) = y(s, y_1, s_1)$, m an integer.

Lemma 2. *Two systems (4) are periodically homeomorphic in the whole space if they have the same number of multipliers, in absolute value less than one, among which there is the same number of negative and*

the same number of negative multipliers, with absolute value greater than one, and have no multipliers with absolute value equal to one.

For the proof of Lemma 2, let us mark the corresponding quantities for the second system (4) by asterisks. Let the principal matrix of the fundamental matrix $Z(s)$ of system (4) be $C = C_1 + C_2 + C_3 + C_4$, where C_i is a $k_i \times k_i$ matrix, and C_1 (C_3) has as its eigenvalues all the negative eigenvalues of the matrix C whose absolute values are less (greater) than one, while C_2 (C_4) has the remaining eigenvalues of the matrix C whose absolute values are less (greater) than one. Construct the matrix $J = -E_{k_1} + E_{k_2} + (-E_{k_3}) + E_{k_4}$. Then there exists a real matrix B such that $\exp B = C^*J$.

Consider the system

$$\frac{dw}{ds} = Bw. \quad (7)$$

We use the homeomorphism, constructed in [4], $w_0^* = \varphi w_0$, to map one system (7) to the other in the plane $s = 0$. In doing so we may choose the homeomorphism φ so that $\varphi J w_0 = J \varphi w_0$. The homeomorphism $(s_1, z_1^*) = \Phi_1(s_1, z_1)$, occurring in the assertion of Lemma 2, is the mapping defined in the plane $s = 0$ by the formula

$$z_0^* = \psi z_0, \quad \psi z_0 = K^*(0) \varphi K^{-1}(0) z_0$$

and continued along trajectories:

$$\begin{aligned} s_1 &= s_1, \\ z_1^* &= z^*(s_1, \psi z(0, z_1, s_1), 0), \end{aligned} \quad (8)$$

where $K(s) = Z(s)W^{-1}(s)J^{-1}(s)$, $W(s)$ is a normal fundamental matrix of system (7), and $J(s)$ is a matrix of such a special form that $J(0) = E_{n-1}$, $J(1) = J$.

Lemma 3. A periodic homeomorphism of two systems (2) in some neighborhood of the s -axis entails a homeomorphism of the corresponding systems (1) in some neighborhood of the closed trajectory.

The mapping Ψ^{-1} (respectively Ψ^{*-1}) assigns to each point x (respectively x^*) of D_2 (respectively D_1^*) a countable set of points $(s+m, y)$ (respectively $(s+m, y^*)$), $m = 0, \pm 1, \pm 2, \dots$, from U_1 (respectively U_1^*). The inverse mapping Ψ (respectively Ψ^*), conversely, maps all points $(s+m, y)$ (respectively $(s+m, y^*)$), m an integer, from U_1 (respectively U_1^*) to a single point x (respectively x^*) of D_1 (respectively D_1^*). The periodic homeomorphism $(s, y^*) = \Phi(s, y)$ appearing in the hypothesis of the lemma puts into correspondence the points $(s+m, y) \in U_1$ and $(s+m, y^*) \in U_1^*$. Therefore $x^* = Tx$, where $T = \Psi^* \Phi \Psi^{-1}$, is a one-to-one mapping of D_1 onto D_1^* . It is not difficult to verify that T and T^{-1} are continuous mappings. Finally, by the very construction of system (2), to the points of one trajectory of system (1) from D_1 , under the mapping Ψ^{-1} , there correspond points of a countable set of trajectories of system (2) from U_1 : $y = y(s+m, y_1, s_1+m)$, m an integer, and under the inverse mapping Ψ , conversely, whence it follows that $x^* = Tx$ is the homeomorphism stated in Lemma 3.

If we now put $\Phi = \Phi_0^{*-1} \Phi_1 \Phi_0$, where Φ_0 is defined by formulas (6), and Φ_1 by formulas (8), then the theorem follows from Lemma 3.

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