ITERATIVE METHODS WITH SECOND-ORDER DIVIDED DIFFERENCES

1.** Let the equation be given

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Abstract

Full Text

MATHEMATICS

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ITERATIVE METHODS WITH SECOND-ORDER DIVIDED DIFFERENCES

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1. Let the equation be given

$$\mathcal{P}(x) = 0, \tag{1}$$

where $\mathcal{P}(x)$ is a nonlinear operator mapping the linear normed space X into a space Y of the same type. In what follows we use the following notation: $\mathcal{P}(x', x'')$, $\mathcal{P}(x', x'', x''')$ are analogues of divided differences ([1]), respectively of the first and second orders, for the operator $\mathcal{P}(x)$; $\Omega_n = [\mathcal{P}(x_n, x_{n-1})]^{-1}$; $U_n = \Omega_n \mathcal{P}(x_n, x_{n-1}, x_{n-2})$; $\tilde{x}_{n+1} - x_n = -\Omega_n \mathcal{P}(x)$; E is the identity operator of the space X.

To introduce a class of iterative methods, suppose that we know three approximations x_n, x_{n-1}, x_{n-2} to the solution x^* of equation (1). We use an analogue of Newton' s interpolation formula ([1]):

$$\begin{gathered}\mathcal{P}(x) = \mathcal{P}(x_n) + \mathcal{P}(x_n, x_{n-1})(x - x_n) + \\ + \mathcal{P}(x_n, x_{n-1}, x_{n-2})(x - x_n)(x - x_{n-1}) + R_n,\end{gathered} \tag{2}$$

where

$$R_n = [\mathcal{P}(x, x_n, x_{n-1}) - \mathcal{P}(x_n, x_{n-1}, x_{n-2})](x - x_n)(x - x_{n-1}). \tag{3}$$

Discarding the remainder term R_n in formula (2), we consider, instead of equation (1), the approximate equation

$$\mathcal{P}(x_n) + \mathcal{P}(x_n, x_{n-1})(x - x_n) + \mathcal{P}(x_n, x_{n-1}, x_{n-2})(x - x_n)(x - x_{n-1}) = 0. \tag{4}$$

Introduce into equation (4) a real parameter α and write this equation in the form

$$\mathcal{P}(x_n) + [\mathcal{P}(x_n, x_{n-1}) - \alpha \mathcal{P}(x_n, x_{n-1}, x_{n-2})(x - x_n)](x - x_n) +$$

$$+(1+\alpha)\mathcal{P}(x_n,x_{n-1},x_{n-2})(x-x_n)^2+$$
$$+\mathcal{P}(x_n,x_{n-1},x_{n-2})(x-x_n)(x_n-x_{n-1})=0. \tag{5}$$

Since, by the chord method ([1]),

$$x^*-x_n \approx \Omega_n\mathcal{P}(x_n)=\tilde{x}_{n+1}-x_n, \tag{6}$$

we replace in equation (5) the element $x-x_n$ in square brackets and in the third and fourth terms by the element $\tilde{x}_{n+1}-x_n$. Then, instead of equation (1), we obtain the approximate linear equation

$$\mathcal{P}(x_n)+[\mathcal{P}(x_n,x_{n-1})-\alpha\mathcal{P}(x_n,x_{n-1},x_{n-2})(\tilde{x}_{n+1}-x_n)](x-x_n)+$$
$$+(1+\alpha)\mathcal{P}(x_n,x_{n-1},x_{n-2})(\tilde{x}_{n+1}-x_n)^2+$$
$$+\mathcal{P}(x_n,x_{n-1},x_{n-2})(\tilde{x}_{n+1}-x_n)(x_n-x_{n-1})=0. \tag{7}$$

Solving equation (7) with respect to x, we take the solution found as the new approximation x_{n+1} to the solution x^* of equation (1). Thus, we arrive at the following class of iterative methods for the approximate solution of equation (1):

$$x_{n+1}=x_n+[E-\alpha U_n(\tilde{x}_{n+1}-x_n)]^{-1}[E-(1+\alpha)U_n(\tilde{x}_{n+1}-x_n)-$$
$$-U_n(x_n-x_{n-1})](\tilde{x}_{n+1}-x_n), \tag{8}$$

where $n=2,3,\dots;$ $x_0,$ $x_1,$ x_2 are three initial approximations to the solution x^* of equation (1).

2. Theorem. Suppose:

1°. Equation (1) has a solution x^*, and

$$\max\{\|x^*-x_0\|,\ \|x^*-x_1\|,\ \|x^*-x_2\|\}\leqslant d.$$

2°. For every $x', x'', x''', x^{\mathrm{IV}}$ from the sphere $\|x-x^*\|\leqslant d$, the estimates hold:

a) $\|[\mathcal{P}(x',x'')]^{-1}\|\leqslant B;$

b) $\|\mathcal{P}(x',x'',x''')\|\leqslant H;$

c) $\|\mathcal{P}(x',x'',x''')-\mathcal{P}(x'',x''',x^{\mathrm{IV}})\|\leqslant K\|x'-x^{\mathrm{IV}}\|.$

3°.

$$|\alpha|BHd+[BK+(1+|\alpha|+|1+\alpha|)B^2H^2]d^2+|1+\alpha|B^3H^3d^3<1.$$

Then the sequence (8) converges to the solution x^* of equation (1) with rate

$$\|x^* - x_n\| \leqslant \frac{1}{M}(Md)^{t_n} \qquad (n = 0, 1, ...), \tag{9}$$

where

$$M = \left[\frac{BK + B^2H^2 + |1+\alpha|B^2H^2(1+BHd)^{7/2}}{1 - |\alpha|BHd(1+BHd)}\right]$$

and the numbers t_n are generalized Fibonacci numbers ($t_0 = t_1 = t_2 = 1;\ t_{n+1} = t_n + t_{n-1} + t_{n-2};\ n = 2, 3, ...$).

Proof. We use the principle of complete induction. On the basis of condition 1°, the estimates (9) are valid for $n = 0, 1, 2$. By formula (8),

$$x^* - x_{n+1} = [E - \alpha U_n(\tilde{x}_{n+1} - x_n)]^{-1}[x^* - \tilde{x}_{n+1} - (1+\alpha)U_n(\tilde{x}_{n+1} - x_n)(x^* - \tilde{x}_{n+1}) + U_n(x^* - x_{n-1})(\tilde{x}_{n+1} - x_n)]. \tag{10}$$

On the basis of Newton' s interpolation formula,

$$x^* - \tilde{x}_{n+1} = -Q_n^* = -U_n(x^* - x_n)(x^* - x_{n-1}) - R_n^*, \tag{11}$$

$$\tilde{x}_{n+1} - x_n = x^* - x_n + Q_n^*, \tag{12}$$

where

$$R_n^* = \Omega_n[\mathcal{P}(x^*, x_n, x_{n-1}) - \mathcal{P}(x_n, x_{n-1}, x_{n-2})](x^* - x_n)(x^* - x_{n-1}), \tag{13}$$

$$Q_n^* = \Omega_n\mathcal{P}(x^*, x_n, x_{n-1})(x^* - x_n)(x^* - x_{n-1}). \tag{14}$$

Replacing in formula (10) $x^* - \tilde{x}_{n+1}$ and $\tilde{x}_{n+1} - x_n$ by formulas (11) and (12), we obtain:

$$x^* - x_{n+1} = [E - \alpha U_n'(x^* - x_n) - \alpha U_{nQ}n^*]^{-1}[-R_n^* + (1+\alpha)U_n(x^* - x_n)Q_n^* + (1+\alpha)U_{nQ}n^{*2} + U_n(x^* - x_{n-1})Q_n^*]. \tag{15}$$

Denote $d_i = \|x^* - x_i\|$. By the Banach theorem,

$$\|[E - \alpha U_n(x^* - x_n) - \alpha U_{nQ}n^*]^{-1}\| \leqslant (1 - |\alpha|BHd - |\alpha|B^2H^2d^2)^{-1} = A(\alpha).$$

Consequently, on the basis of formula (15),

$$d_{n+1} \leqslant M^{-3}\big[M_1^2(Md)^{t_n+t_{n-1}+t_{n-2}} + M_2^2(Md)^{2t_n+t_{n-1}} + M_3^2(Md)^{t_n+2t_{n-1}}\big], \quad (16)$$

where

$$M_1^2 = BKA(\alpha), \qquad M_2^2 = [1+\alpha]B^2H^2(1+BHd)A(\alpha), \qquad M_3^2 = B^2H^2A(\alpha).$$

Since $Md < 1$ (condition 3°) and $2t_n + t_{n-1} \geqslant t_n + 2t_{n-1} \geqslant t_n + t_{n-1} + t_{n-2}$, it follows that

$$d_{n+1} \leqslant \frac{1}{M}(Md)^{t_{n+1}}. \quad (17)$$

Passing to the limit in formula (17) as $(n \to \infty)$, we find that $x_n \to x^*$. The theorem is proved.

Remark. The sphere $\|x^* - x\| \leqslant d$ in condition 2° of the theorem may be replaced, for example, by the sphere $\|x - x_0\| \leqslant 2d$. Indeed, if x is an element of the first sphere, then

$$\|x - x_0\| \leqslant \|x - x^*\| + \|x^* - x_0\| \leqslant 2d,$$

i.e., x also belongs to the second sphere. If the estimates 2° a, b, c are found in the sphere $\|x - x_0\| \leqslant 2d$, then it is easy to see that the solution x^* is unique in the sphere $\|x - x_0\| \leqslant d$.

3. Let us note that the class of iterative methods (8) is an interpolation analogue of the class of differential methods studied in the works of R. Ludwig ([2]) and Yu. Ya. Kaazik ([3]). From the estimates (9) it is seen that the methods of class (8) converge to the solution of equation (1) faster than the chord method, for which the exponents t_n are the ordinary Fibonacci numbers ([1]). The methods (8) are effective for solving such nonlinear transcendental equations for which finding derivatives of the functions is difficult. For the approximate solution of algebraic and transcendental equations

$$f(x) = 0 \quad (18)$$

the methods (8) have the form:

$$x_{n+1} = x_n + \{[f(x_n, x_{n-1}) - (1+\alpha)f(x_n, x_{n-1}, x_{n-2})(\tilde{x}_{n+1} - x_n) -$$

$$-f(x_n, x_{n-1}, x_{n-2})(x_n - x_{n-1})]/[f(x_n, x_{n-1}) -$$
$$-\alpha f(x_n, x_{n-1}, x_{n-2})(\tilde{x}_{n+1} - x_n)]\}(\tilde{x}_{n+1} - x_n), \quad (19)$$

where $\tilde{x}_{n+1} - x_n = -f(x_n)/f(x_n, x_{n-1})$, $n = 2, 3, \ldots$.

Taking in formula (8) (or (19)) $\alpha = 0$ and $\alpha = -1$, we obtain the iterative methods considered by the author ([4]) and which are, respectively, interpolation analogues of the methods of tangent parabolas ([5]) and tangent hyperbolas ([6]).

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Note: Figure translations are in progress. See original paper for figures.