A PROBLEM OF A DISTRIBUTOR OPTIMIZING ON AN INTERVAL

1964

SovietRxiv

Abstract

Full Text

CYBERNETICS AND CONTROL THEORY

A. A. BLYUSIN

A PROBLEM OF A DISTRIBUTOR OPTIMIZING ON AN INTERVAL

(Presented by Academician L. S. Pontryagin, 29 II 1964)

In this note we consider certain problems on the optimal distribution of a given flow $q(t)$ entering a certain system and distributed by it among n directions. Such a distribution of the flow is described mathematically by specifying n functions of time $u_i(t) \geqslant 0$, $i = 1, 2, \dots, n^*$, subject to the condition: $\sum u_i(t) = 1$. ($u_i(t)$ expresses the fraction of the flow going to the i-th direction at time t.) In the more general case, instead of the inequality $u_i(t) \geqslant 0$ it may be required that the $u_i(t)$ take values in certain prescribed intervals $[u_{i0}(t), u_{i1}(t)]$. We shall call a collection of functions $u(t) = \{u_1(t), \dots, u_n(t)\}$ a **distributor** (although in the case where negative $u_i(t)$ are allowed, this name is rather conventional). Suppose it is required to find such a distributor that would minimize the functional $\Phi_u(T) = \sum f_i(T, x_i(T))$, where the $f_i(t, x_i)$ are differentiable in t and in x_i almost everywhere,

$$x_i(t) = \int_0^t q(\tau) u_i(\tau)\, d\tau,$$

and $x_i(t)$ may take values in certain prescribed intervals $[x_{i0}(t), x_{i1}(t)]$.

The solution of this problem, generally speaking, is nonunique. This may be interpreted in such a way that additional conditions can be imposed on the desired optimal distributor $u^*(t)$. In some problems of flow distribution it is natural to impose the requirement that one and the same distributor $u^*(t)$ be optimal in the minimization of each functional $\Phi_u(T)$, where $T \in [T_0, T_1]$**. Such a distributor will be called** optimizing on the interval** $[T_0, T_1]$. Analogously, one can define a distributor **optimizing on a certain set** of values of t.

Let us consider the space of phase trajectories $\{x_i(t)\}$. Then it can be proved that, if $u^*(t)$ optimizes on the interval $[T_0, T_1]$, there exist functions $\lambda(T)$ and $\alpha_i(T)$, $0 \leqslant \alpha_i(T) \leqslant 1$, such that at the points of the corresponding phase trajectory $\{x_i^*(T)\}$ the following conditions are satisfied:

$$\frac{\partial f_i^-}{\partial x_i} + \alpha_i \left(\frac{\partial f_i^+}{\partial x_i} - \frac{\partial f_i^-}{\partial x_i} \right) = \lambda(T), \qquad X_{i0}(T) < x_i^*(T) < X_{i1}(T); \qquad (1)$$

$$\frac{\partial f_i^+}{\partial x_i} \geqslant \lambda(T), \qquad x_i^*(T) = X_{i0}(T); \tag{2}$$

$$\frac{\partial f_i^-}{\partial x_i} \leqslant \lambda(T), \qquad x_i^*(T) = X_{i1}(T), \tag{3}$$

where

$$f_i^+ = f_i(T, x_i^* + 0), \qquad f_i^- = f_i(T, x_i^* - 0),$$

$$X_{i0}(T) = \max\left\{ x_{i0}(T), \int_0^T q(\tau)\left[U_{i0}(\tau)\theta(q) + U_{i1}(\tau)\theta(-q)\right] d\tau \right\};$$

$$X_{i1}(T) = \min\left\{ x_{i1}(T) \int_0^T q(\tau)\left[U_{i0}(\tau)\theta(-q) + U_{i1}(\tau)\theta(q)\right] d\tau \right\};$$

$$U_{i0}(t) = \max\left\{ u_{i0}(t),\ 1 - \sum_{j \neq i} u_{j1}(t) \right\};$$

$$U_{i1}(t) = \max\left\{ u_{i1}(t),\ 1 - \sum_{j \neq i} u_{j0}(t) \right\}; \qquad \theta(q) = \begin{cases} 1, & q(t) > 0, \\ 1, & q(t) \leq 0. \end{cases}$$

If $\partial^2 f_i / \partial x_i^2 > 0,\ x_i(T) \in [X_{i0}(T),\ X_{i1}(T)]$, then the listed conditions are also sufficient, and the distributor $u^*(T)$ is unique. If, in addition, $\partial^2 f_i / \partial x_i \partial t \equiv 0,\ u_{i0} \equiv 0,\ u_{i1} \equiv 1,\ \dot{x}_{i0} \equiv \dot{x}_{i1} \equiv 0$, then one can assert that an optimizing distributor exists everywhere.*

Solving the system of ordinary differential equations

$$\dot{x}_i(t) = \begin{cases} h_i \left(q - \sum\limits_{N_0(t)} \dot{X}_{j0} - \sum\limits_{N_1(t)} \dot{X}_{j1} + \sum\limits_{N(t)} h_j \dfrac{\partial^2 (f_j - f_i)}{\partial x_j \partial t} \right) \left(\sum\limits_{N(t)} h_j \right)^{-1}, & i \in N(t), \\ \dot{X}_{i0}, & i \in N_0(t), \\ \dot{X}_{i1}, & i \in N_1(t), \\ 0, & i \in M(t), \end{cases}$$

---

* Here and below $i = 1, \ldots, n$.

** In what follows, unless otherwise specified, we shall understand $T \in [T_0, T_1]$.

where **

$$N(t) = \left\{ j : \frac{\partial f_j^-}{\partial x_j} = \lambda(t) \text{ or } \frac{\partial f_j^+}{\partial x_j} = \lambda(t) \right\}, \qquad M(t) = \{ j : X_{j0} < x_j(t) < X_{j1}(t),\ 0 < \alpha_j(t) < 1 \},$$

$$N_0(t) = \left\{ j : \frac{\partial f_j^+}{\partial x_j} > \lambda(t),\ x_j(t) = X_{j0}(t) \right\}, \qquad N_1(t) = \left\{ j : \frac{\partial f_j^-}{\partial x_j} < \lambda(t),\ x_j(t) = X_{j1}(t) \right\},$$

$$h_j = \left(\frac{\partial^2 f_j}{\partial x_j^2} \right)^{-1},$$

with the initial conditions $x_i(0) = 0$, we obtain a certain phase trajectory $\{x_i^{**}(t)\}$. It has the property that, if

$$q(t_0) u_{i0}(t_0) \le \dot{x}_i^{**}(t_0) \le q(t_0) u_{i1}(t_0), \tag{4}$$

then t_0 may be regarded as T_0 up to t_1, when (4) ceases to hold, and $[q(t)]^{-1} \dot{x}_i^{**}(t)$, $q(t) \ne 0$, $t_0 \le t \le t_1$, as $u^*(T)$, $T \in [t_0, t_1]$. With its help one can also obtain the optimal distributor $u^*(t)$, $0 \le t < T_0$, satisfying the condition $\Phi_{u^*}(T_0) = \min_{u(t)} \Phi_u(T_0)$.

When there is a mutually one-to-one dependence

$$s = \int_0^t q(\tau)\, d\tau$$

it is not difficult to obtain the corresponding distributor $v^*(s) = u^*(t)$, regarding s as the new independent variable. In general, however, in this case it is easier to obtain the optimizing distributor $v^*(s)$ directly in parametric form, usually taking the quantity λ as the parameter.

In the case when, for example, $\partial f_i / \partial t \equiv 0$, $\dot{f}_i > 0$, $u_{i0} = x_{i0} = 0$, $u_{i1} = x_{i1} = \infty$, the everywhere-optimizing distributor (the unique

[*] Everything said above remains valid also in the case where, instead of $f_i(t, x_i)$, one considers $f(t, x_1, \dots, x_n)$. In this case conditions (1)–(3) will be necessary and sufficient if

$$\sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j} \xi_i \xi_j > 0, \qquad \xi_i, \xi_j \ne 0.$$

** If $j \in N$, then the derivatives in the equations are taken from the right or from the left depending on the side from which the corresponding equality is satisfied.

everywhere) is obtained simply:

$$v_i^*(\lambda) = \begin{cases} \dot{\varphi}_i(\lambda) \left[\sum\limits_{N(\lambda)} \dot{\varphi}_j(\lambda) \right]^{-1}, & i \in N(\lambda), \\ 0, & i \in \overline{N}(\lambda), \end{cases}$$

$$s(\lambda) = \sum_{N(\lambda)+M(\lambda)} \varphi_j(\lambda),$$

where $\varphi_i(\lambda)$ is the real root of equation (1) for fixed λ. When the values of λ run from $\min_i \dot{f}_i(0)$ to $\min_i \lim_{x_i \to \infty} f(x_i)$, then s varies from 0 to ∞. In this case conditions (1)–(3), obviously, are satisfied with observance of the constraints imposed on the distributor and on the phase coordinates.

In particular, when $f_i(x_i) = c_i \exp(-r_i^{-1} x_i)$, where c_i, r_i are positive constants and the numbering is chosen so that $c_i/r_i \geq c_{i+1}/r_{i+1}$, then the optimizing distributor everywhere has the form:

$$v_i^*(k) = \begin{cases} \dfrac{r_i}{R_k}, & 1 \leq i \leq k, \\ 0, & i > k, \end{cases}$$

$$\sum_1^{k-1} \Delta_m \leq s(k) < \sum_1^{k} \Delta_m,$$

where

$$\Delta_m = R_m \ln \frac{c_i r_{i+1}}{c_{i+1} r_i}, \qquad R_m = \sum_1^m r_i.$$

Here, in the role of the parameter it was more convenient to take the number of the direction for which the condition

$$\dot{f}_k(0) = \max_j \dot{f}_j(0), \qquad j \in N(\lambda).$$

is satisfied. With the aid of this same parameter one can also obtain the dependence $\Phi_{v^*}(s)$:

$$\Phi_{v^*}(k) = R_k \exp\left[\frac{1}{R_k}\left(\sum_1^k r_i \ln\frac{c_i}{r_i} - s(k)\right)\right].$$

In conclusion we make the following further remarks on certain properties of the distributors $u^*(t)$ optimizing on the interval $[T_0, T_1]$, or of the corresponding distributors $v^*(s)$.

Remark 1.

$$\int_{T_0}^{T} F(\tau, \Phi_{u^*}(\tau))\, d\tau = \min_{u(t)} \int_{T_0}^{T} F(\tau, \Phi_u(\tau))\, d\tau, \qquad 0 \le t \le T,$$

where $\partial F(T,z)/\partial z \ge 0$. Hence, in particular, considering $F(t,z) \equiv z$ and taking into account that for $u_{i0}(t) \equiv -\infty$, $u_{i1}(t) \equiv \infty$ there exists everywhere an optimizing distributor, one may assert:

$$\bar{u}^*(t) = \{u_i^* = x_i^*(t)[s(t)]^{-1}\}$$

is everywhere optimizing for

$$\Phi_{\bar{u}}(T) = \sum_i \int_0^T \bar{f}_i(\tau, \bar{u}_i(\tau))\, d\tau,$$

where

$$\bar{f}_i(t,z) = f_i(t, s(t)z),$$

and the constraints imposed on $\bar{u}_i(t)$ are

$$\sum_i \bar{u}_i(t) = 1, \qquad x_{i0}(t)[s(t)]^{-1} \le \bar{u}_i(t) \le x_{i1}(t)[s(t)]^{-1}.$$

Remark 2. If $u_{i0}(t), x_{i0}(t), \partial f_i/\partial t \equiv 0$, $u_{i1}(t), x_{i1}(t) \equiv \infty$, $-\dot{f}_i, \ddot{f}_i, \dot{f}_i$, $-\partial Q(t,z)/\partial z > 0$, then the functional with "feedback"

$$\hat{\Phi}_u(T) = \sum_i f_i\left(\int_0^T Q(\tau, \hat{\Phi}_u) u_i\, d\tau\right)$$

is minimized everywhere by the distributor $v^*(s)$, $0 \le s < \infty$. Moreover, the latter also possesses the property of optimality with respect to speed of response: it ensures the equality $\hat{\Phi}_u(T) = c$, where

$c < \sum f_i(0)$, in minimum time. If, however, $\partial Q(t,z)/\partial z > 0$, $\dot{f}_i > 0$, and $c > \sum f_i(0)$, then $v^*(s)$ ensures optimality with respect to "duration of action." Under the conditions considered, $v^*(s)$ is unique in every sense.

Remark 3. If among the $f_i(t, x_i)$ there is an $f_{i_0}(t, x_{i_0})$ and $u_{i_0 0} \equiv x_{i_0 0} \equiv -\infty$, then the distributor optimizing on $[T_0, T_1]$ is correspondingly also optimizing under the condition $\sum u_i(t) \geqslant 1$ instead of $\sum u_i(t) = 1$.

For $u_{i_0 1} = x_{i_0 1} = \infty$, the same is true for $\sum u_i(t) \leqslant 1$. With a combination of the conditions, evidently, no restrictions need be imposed on $\sum u_i(t)$.

Received
13 II 1964

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.