CYBERNETICS AND CONTROL THEORY

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Abstract

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CYBERNETICS AND CONTROL THEORY

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ON SELF-CORRECTING GATE CIRCUITS

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In this note self-correcting ([1]) gate circuits are considered. In gate circuits two kinds of malfunctions are possible: shorting, or breakdown, and disconnection of gates. Below estimates are given for the complexity of gate circuits of depth 2 ([2]) that correct disconnections of gates. It is shown that the complexity of circuits correcting m-fold disconnections increases, in comparison with the complexity of ordinary circuits, by no less than $C(m+1)$ times, where C is a certain constant, and in the case of not too "dense" matrices—asymptotically by $m+1$ times. Thus the case considered here differs substantially from the case of correction of a single shorting in contact circuits ([1]), where no (asymptotic) increase of complexity is required in comparison with ordinary circuits.

1°. Denote by $\chi(\mathfrak{A})$ the matrix realized by the gate circuit $\mathfrak{A}$. Denote by $\mathfrak{M}_m(\mathfrak{A})$ the set of all circuits obtained from the circuit $\mathfrak{A}$ by disconnecting no more than m arbitrary gates. We shall say that the circuit $\mathfrak{A}$ corrects m-fold disconnections if for every circuit $\mathfrak{C}$ from $\mathfrak{M}_m(\mathfrak{A})$ the equality $\chi(\mathfrak{C}) = \chi(\mathfrak{A})$ holds.

Denote by $B(\mathfrak{A})$ the number of gates in the circuit $\mathfrak{A}$. For an arbitrary matrix A* introduce the function $B_{r,m}(A) = \min B(\mathfrak{A})$, where the minimum is taken over all circuits $\mathfrak{A}$ of depth no greater than r, realizing the matrix A and correcting m-fold disconnections. For an arbitrary class of matrices $\mathfrak{N}$ introduce the function $B_{r,m}(\mathfrak{N}) = \max B_{r,m}(A)$, where the maximum is taken over all matrices A from $\mathfrak{N}$. The functions $B_{r,0}$ refer to ordinary circuits without self-correction. We shall call the number

$$\beta_{r,m}(\mathfrak{N}) = \frac{B_{r,m}(\mathfrak{N})}{B_{r,0}(\mathfrak{N})}$$

the duplication coefficient.

Obviously, for any $r, \mathfrak{N}$, $\beta_{r,m}(\mathfrak{N}) \leq m+1$.

Lemma 1. *In order that a circuit correct m-fold disconnections, it is necessary and sufficient that in it, between any pair of poles with nonzero conductivity, there exist no fewer than $m+1$ distinct chains having no pairwise common gates.*

Proof follows from the "transport" theorem of Ford–Fulkerson ([3]).

Fig. 1

Figure 1: Fig. 1

Denote by $\|A\|$ the number of ones in the matrix A. We shall call a matrix having s rows and t columns an (s,t)-matrix; an (s,t)-matrix will be called nondegenerate if $s>0,\ t>0$; a nondegenerate matrix consisting only of ones will be called complete. We shall call a circuit $\mathfrak{A}$ nondegenerate if $\chi(\mathfrak{A})$ is a nondegenerate nonzero matrix. Let a be a nondegenerate (s,t)-matrix, A a nondegenerate nonzero matrix. Introduce the densities of matrices:

$$d(a)=d(s,t)=\begin{cases}\dfrac{st}{s+t}, & \text{for } s>1,\ t>1,\\ 1, & \text{for } s=1 \text{ or } t=1;\end{cases}$$

$$\delta(A)=\max d(a),$$

where the maximum is taken over all complete submatrices a of the matrix A;

$$\delta(\mathfrak{N})=\min_{A\in\mathfrak{N}}\delta(A).$$

Theorem 1. *If A is a nondegenerate nonzero matrix, then*

$$(m+1)\frac{\|A\|}{\delta(A)}\leq B_{2,m}(A).$$

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* All matrices are considered Boolean.

Proof. Let $B_{2,m}(A)$ be attained on a nondegenerate (p,q)-circuit $\mathfrak{A}$ of depth at most 2. Obviously, $\mathfrak{A}$ has no "superfluous" gates. We decompose $\mathfrak{A}$ into subcircuits $\mathfrak{A}_k$, pairwise intersecting only at poles, so that $\mathfrak{A}_0$ is a circuit of depth at most 1, while $\mathfrak{A}_k$ for $k\geqslant 1$ are circuits of depth 2 containing only one internal node (Fig. 1). We assume that every subcircuit $\mathfrak{A}_k$ contains all the poles of the circuit $\mathfrak{A}$. Denote by $\varphi_k^{i,j}$ the largest number of chains containing no pairwise common gates (a flow) between the poles with numbers i,j in the subcircuit $\mathfrak{A}_k$. Every chain of the circuit $\mathfrak{A}$ belongs to one and only one subcircuit $\mathfrak{A}_k$; therefore (Lemma 1)

Fig. 1

$$(m+1)\|A\|\leqslant\sum_k\sum_{i,j}\varphi_k^{i,j}. \tag{1}$$

We shall show that, for nondegenerate subcircuits $\mathfrak{A}_k$,

$$\sum_{i,j} \varphi_k^{i,j} \leqslant \delta(\chi(\mathfrak{A}_k))\, B(\mathfrak{A}_k). \tag{2}$$

For $k = 0$, (2) is obvious. Let $k \geqslant 1$. Denote by a_i and b_j (we omit the index k) the number of gates connecting the internal node of the subcircuit $\mathfrak{A}_k$ with the i-th input and the j-th output, respectively. Obviously, $\varphi^{i,j} = \min(a_i, b_j)$. Let s be the number of inputs and t the number of outputs connected with the internal node of the subcircuit $\mathfrak{A}_k$. Renumber the nonzero a_i, b_j by the numbers $u = 1, \dots, s$, $v = 1, \dots, t$. Since

$$\frac{st}{s+t} \leqslant \delta(\chi(\mathfrak{A}_k)), \qquad B(\mathfrak{A}_k) = \sum_{u=1}^{s} a_u + \sum_{v=1}^{t} b_v,$$

(2) follows from the easily verified inequality

$$\sum_{u=1}^{s} \sum_{v=1}^{t} \min(a_u, b_v) \leqslant \frac{st}{s+t} \left(\sum_{u=1}^{s} a_u + \sum_{v=1}^{t} b_v \right).$$

The assertion of the theorem follows from (1), (2), since

$$\max \delta(\chi(\mathfrak{A}_k)) \leqslant \delta(A), \qquad B_{2,m}(A) = B(\mathfrak{A}) = \sum_k B(\mathfrak{A}_k).$$

2°. Denote by $\mathfrak{B}(p, q, \alpha)$ the class of all (p, q)-matrices containing αpq ones (the conditions $0 \leqslant \alpha \leqslant 1$ and that αpq is an integer are assumed to hold everywhere, without further mention). Put $\alpha^* = \min(\alpha, 1-\alpha)$ and

$$H(z) = z \lg_2 \frac{1}{z} + (1-z) \lg_2 \frac{1}{1-z}. \tag{3}$$

Theorem 2. Suppose the following conditions are satisfied:

a) $q_n \leqslant p_n$;

b) $q_n \to \infty$;

c)
$$\frac{\alpha_n \lg_2 \frac{1}{\alpha_n} q_n}{\lg_2 p_n} \to x;$$

d)
$$\frac{\lg_2 p_n}{\lg_2 \frac{1}{\alpha_n^*}} \to \infty \ * .$$

* Conditions c), d) mean that the matrices from the classes $\mathfrak{B}(p_n, q_n, \alpha_n)$ are not too "narrow" and not too "sparse" or "dense."

Then

$$\delta(\mathfrak{B}(p_n, q_n, \alpha_n)) \sim \frac{\lg_2 p_n}{\lg_2 \frac{1}{\alpha_n}}.$$

Proof. The lower bound follows from the existence in every matrix from $\mathfrak{B}(p_n, q_n, \alpha_n)$ of a complete (s_n, t_n)-submatrix such that

$$t_n \sim \frac{\lg_2 p_n}{\lg_2 \frac{1}{\alpha_n}}, \qquad \frac{s_n}{t_n} \to \infty \tag{4}$$

(the numbers s_n, t_n are the same for all matrices from $\mathfrak{B}(p_n, q_n, \alpha_n)$). Indeed, from (4)

$$\delta(\mathfrak{B}(p_n, q_n, \alpha_n)) \geq \frac{s_n t_n}{s_n + t_n} \sim \frac{\lg_2 p_n}{\lg_2 \frac{1}{\alpha_n}}.$$

As in (4), it is easy to show that for every t_n, not greater than $[\alpha_n q_n]$, in every matrix from $\mathfrak{B}(p_n, q_n, \alpha_n)$ there exists a complete (s_n, t_n)-submatrix, where

$$s_n = \left] p_n \frac{C_{[\alpha_n q_n]}^{t_n}}{C_{q_n}^{t_n}} \right[* . \tag{5}$$

We shall show that for a certain choice of the number t_n the pair (s_n, t_n) satisfies the conditions (4). Put

$$t_n \to \infty, \qquad \frac{t_n}{\alpha_n q_n} \to 0. \tag{6}$$

In view of c), d) these conditions are compatible and follow from (4). From (5)

$$\lg_2 s_n \geq \lg_2 p_n + H\left(\frac{t_n}{\alpha_n q_n}\right) \alpha_n q_n - H\left(\frac{t_n}{q_n}\right) q_n + o(1) = \lg_2 p_n - t_n \lg_2 \frac{1}{\alpha_n} -$$

$$-(\alpha_n q_n - t_n) \lg_2 \left(1 - \frac{t_n}{\alpha_n q_n}\right) + (q_n - t_n) \lg_2 \left(1 - \frac{t_n}{q_n}\right) + o(1) =$$

$$= \lg_2 p_n - t_n \lg_2 \frac{1}{\alpha_n} - \frac{1-\alpha_n}{\alpha_n} \frac{t_n^2}{q_n} (1+o(1)) \lg_2 e + o(1) \ \ **.$$

Introduce a parameter ε_n and put

$$t_n = \left[(1-\varepsilon_n) \frac{\lg_2 p_n}{\lg_2 \dfrac{1}{\alpha_n}} \right].$$

Then

$$\lg_2 s_n \geq \varepsilon_n \lg_2 p_n - \frac{1-\alpha_n}{\alpha_n} \frac{t_n^2}{q_n} (1+o(1)) \lg_2 e + o(1).$$

Put

$$\lambda_n = \frac{\lg_2 p_n}{\lg_2 \dfrac{1}{\alpha_n}}, \qquad \omega_n = \frac{\lg_2 \dfrac{1}{\alpha_n} q_n}{\lg_2 p_n}, \qquad \varphi_n = \frac{\lg_2 p_n}{\lg_2 \lambda_n}.$$

In view of d), $\varphi_n \to \infty$. The conditions (4) follow from the conditions $\varepsilon_n \to 0$, $\varepsilon_n \omega_n \to \infty$ $***$, $\varepsilon_n \varphi_n \to \infty$. Put

$$\varepsilon_n = (\min(\omega_n, \varphi_n))^{-1/2}.$$

Then all the conditions are fulfilled.

* $[a]$ ($]a[$) denotes the integer nearest to a from below (from above).

** The following facts are used: Stirling' s formula, (6), $H(z)$ increases monotonically in a neighborhood of zero, $H(0) = 0$, (3), $\lg_2(1-z) = -z(1+o(1)) \lg_2 e$ as $z \to 0$.

*** $\varepsilon_n \lg_2 p_n \dfrac{\alpha_n q_n}{(1-\alpha_n) t_n^2} \asymp \varepsilon_n \omega_n \dfrac{\lg_2 \dfrac{1}{\alpha_n}}{1-\alpha_n} \to \infty$, since $\dfrac{\lg_2 \dfrac{1}{\alpha_n}}{1-\alpha_n} \geq 1$.

Upper estimate. Denote by $S(p, q, \alpha, s, t)$ the number of matrices from $\mathfrak{B}(p, q, \alpha)$ having at least one complete (s, t)-submatrix. We have

$$S(p,q,\alpha,s,t) \le C_p^s C_q^t C_{pq-st}^{\alpha pq-st}.$$

Denote by $\varkappa_n$ the cardinality of the class $\mathfrak{B}(p_n,q_n,\alpha_n)$. Put $\nu_n = \dfrac{\lg_2 q_n}{\lg_2 p_n}$. Under conditions a), b) and $s_{nt}n = o(\alpha_n p_n q_n)$, we have

$$\lg_2 \frac{S(p_n,q_n,\alpha_n,s_n,t_n)}{\varkappa_n} \le (s_n+\nu_n t_n)\lg_2 p_n - s_{nt}n \lg_2 \frac{1}{\alpha_n}.^* \tag{7}$$

Suppose the upper estimate is false. Then there exist a number ω, greater than one, and a subsequence of $\{n\}$ (for it we retain the notation $\{n\}$) such that for every matrix A_n from $\mathfrak{B}(p_n,q_n,\alpha_n)$, $\omega\lambda_n \le \delta(A_n)$, i.e., every matrix A_n contains a complete (u_n,v_n)-submatrix a_n (u_n, v_n depend on A_n) for which $\omega\lambda_n \le d(a_n)$. Every matrix a_n contains a complete (s_n,t_n)-submatrix b_n (s_n, t_n depend on a_n) such that $s_{nt}n = o(\alpha_n p_{nq} n)$ and $\omega\lambda_n \le d(b_n)$. Indeed, without loss of generality assume that $v_n \le u_n$. If $v_n \le 2\omega\lambda_n$, put $(s_n,t_n)=(u_n,v_n)$; if $2\omega\lambda_n < v_n$, put $s_n = t_n =]2\omega\lambda_n[$. Condition $s_{nt}n = o(\alpha_{np_{nq}}n)$ follows from c). But when $\omega\lambda_n \le d(b_n)$, the right-hand side of (7) tends to $-\infty$, i.e., as n grows, almost all matrices from $\mathfrak{B}(p_n,q_n,\alpha_n)$ contain no complete (s_n,t_n)-submatrices. The contradiction obtained completes the proof of the theorem.

Corollary. Under the conditions of Theorem 2,

$$\frac{\alpha_n \lg_2 \frac{1}{\alpha_n}}{H(\alpha_n)}(m+1) \le \beta_{2,m}(\mathfrak{B}(p_n,q_n,\alpha_n)) \le m+1.$$

Indeed, from (5),

$$B_{2,0}(\mathfrak{B}(p_n,q_n,\alpha_n)) \sim H(\alpha_n)\frac{p_{nq}n}{\lg_2 p_n}.$$

Theorem 3. Under the conditions of Theorem 2 and $\alpha_n \to 0$,

$$\beta_{2,m}(\mathfrak{B}(p_n,q_n,\alpha_n)) \sim m+1.$$

Theorem 4. If $\min(p_n,q_n)\to\infty$, then for the class of all (p_n,q_n)-matrices

$$\frac{m+1}{e\ln 2} \le \beta_{2,m} \le m+1.$$

Proof. From the corollary,

$$(m+1)\max\left(\alpha, \lg_2 \frac{1}{\alpha}\right) \leq \beta_{2,m}.$$

The maximum is attained at $\alpha = \dfrac{1}{e}$.

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CITED LITERATURE

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[2] O. B. Lupanov, *DAN*, **111**, No. 6 (1956).
[3] C. Berge, *Graph Theory and Its Applications*, IL, 1962, p. 87.
[4] E. I. Nechiporuk, *DAN*, **148**, No. 1 (1963).
[5] E. I. Nechiporuk, in: *Problems of Cybernetics*, vol. 9, 1963.

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* $x_n \leqslant\cdot\, y_n$ means that $x_n \leqslant y_n$ for sufficiently large n.

The following facts are used: Stirling' s formula,

$$H'(z) = \lg_2 \frac{1-z}{z}, \qquad H''(z) < 0 \quad \text{for } 0 < z < 1,$$

$$H\left(\frac{\alpha_{np_{nq}} n - s_{nt} n}{p_{nq} n - s_{nt} n}\right) = H\left(\alpha_n - \frac{(1-\alpha_n) s_{nt} n / p_{nq} n}{1 - s_{nt} n / p_{nq} n}\right) \leqslant$$

$$\leqslant H(\alpha_n) - \frac{(1-\alpha_n) s_{nt} n / p_{nq} n}{1 - s_{nt} n / p_{nq} n} \lg_2 \frac{1-\alpha_n}{\alpha_n}.$$

Note: Figure translations are in progress. See original paper for figures.