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Abstract

Full Text

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SOLUTIONS OF THE WORD PROBLEM FOR ONE CLASS OF GROUPS BY MEANS OF DEHN' S ALGORITHM, AND OF THE CONJUGACY PROBLEM BY MEANS OF ONE GENERALIZATION OF DEHN' S ALGORITHM

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§ 1. Let a group G be given by generating elements $a_1, \dots, a_n$ and defining relations $R_1 = 1, \dots, R_k = 1$, where: 1) each word R_i is reduced; 2) the set of words $\{R_i\}$ is closed under the operations of taking inverses and taking cyclic permutations of the letters of the words R_i; 3) if R_i and R_j are not mutual inverses, then $< \frac{1}{4}$ of the letters of the word R_i are cancelled in reducing the product $R_{iR}j$; 4) if each of the words R_i, R_j, and R_k is written on one side of a triangle, then cancellation cannot occur at all three vertices.

The algorithm solving the word problem for the group G is as follows.

An arbitrary word W is given. In order to find out whether W is equal to the identity, we apply to the word W, as long as this is possible, the following two operations: α) reduction; β) replacement of S by T, if $R_i \simeq S\overline{T}$ and $l(S) > l(T)$. (Here and below $\simeq$, $=$, and $\equiv$ denote, respectively, graphical equality, equality in the free group, and equality in the group G; $l(a_{i_1}^{\varepsilon_1} a_{i_2}^{\varepsilon_2} \cdots a_{i_n}^{\varepsilon_n}) = n$, and $\overline{T}$ denotes T^{-1}.)

By virtue of Theorem 2, $W = 1$ if and only if this process ends with the empty word. This algorithm was first applied by M. Dehn to another class of groups ([1]).

The conjugacy problem for the group G is solved in the following way. Let arbitrary two words X and Y be given. To find out whether they are conjugate, i.e. whether there exists a word Z such that $X = \overline{Z}YZ$, we write the word X on a circle and apply, as long as this is possible, the operations α) and β). Cutting the transformed circular word at all possible places, we obtain a sequence of words $X_1, \dots, X_j, \dots, X_\alpha$. If $R_i \simeq \overline{A}X_{jA}\overline{Q}_j$, $l(Q_j) \leq \frac{1}{2}l(R_i)$, and all cyclic permutations $Q_{j1}, \dots, Q_{j\beta}$ of the word Q_j are reduced, then we add them to the list of words $X_1, \dots, X_\alpha$. Then, with the help of the word Q_{jk}, we obtain new words in the list in the same way as from the word X_j, and so on.

Let us call the words in the final list $X_1, \dots, X_p$. From the word Y we obtain in an analogous way a list of words $Y_1, \dots, Y_q$. For each word Z such that $l(Z) \leq \frac{1}{2}\max_{1\leq i\leq k} l(R_i)$, and for each pair of words (X_i, Y_j), we determine whether the equality

$$X_i = \overline{Z} Y_{jZ}$$

holds.

Theorem 1. *X and Y are conjugate if and only if at least one of this finite number of equalities holds.*

This algorithm is a generalization of the algorithm which M. Dehn invented for solving the conjugacy problem for another class of groups ([1]).

§ 2. In order to formulate Theorem 2, it is first necessary to define the notion of an "adjacent n-gon."

Definition 1. If, under the given method of cancellation (complete or partial) of the word $\prod_{i=1}^{m} \overline{T}_i R_{iT} i$, the words R_{ij} cancel with the words $R_{i_{j+1}}$ $(j = 1, 2, \dots, n-1;\ 1 \leqslant i_1 < i_2 < \dots < i_n \leqslant m)$, no R_{i_p} $(1 \leqslant p \leqslant n)$ cancels with any $\overline{T}_i$ or T_i, except for the indicated cancellations, cancels with no more than one R_i among one or two R_{iq}, R_{ip}, and if, after cancellation, subwords $R^c_{i_p}$ $(1 \leqslant p \leqslant n)$ remain from the words R_{i_p}, then the word $R^c_{i_1} R^c_{i_2} \dots R^c_{i_n}$ is called an **adjacent** n-tuple.

Theorem 2. *If an irreducible nonempty word W is equal to the identity in the group G, then there exists a word $\prod_{i=1}^{m} \overline{T}_{iR_{iT}} i$ such that*

$$W \equiv \prod_{i=1}^{m} \overline{T}_{iR_{iT}} i$$

and, for any method of cancellation of the word $\prod_{i=1}^{m} \overline{T}_{iR_{iT}} i$, an adjacent n-tuple remains.

To prove Theorem 2, products $\prod_{i=1}^{m} \overline{T}_{iR_{iT}} i$ are considered such that R_i absorbs $< \frac{1}{4}$ of the letters of the word R_j for all i and j, and $\leqslant 2R_i$ cancel with R_j on each side under any method of cancellation; each word $\overline{T}_{iR_{iT}} i$ is irreducible, and neither $\overline{T}_i$ nor T_i contains $> \frac{1}{2} R_j$.

For such products the following notation is introduced for infinite sequences of assertions.

A_q. If $m \leqslant q$ and $j < k$, then $\overline{T}_j$ does not cancel with R_k, and T_k does not cancel with R_j, under any method of cancellation of the word $\prod_{i=1}^{m} \overline{T}_{iR_{iT}} i$.

B_q. If $m \leqslant q$ and $j < k < p$, then T_j and T_k do not cancel with R_p, and $\overline{T}_k$ and $\overline{T}_p$ do not cancel with R_j under any method of cancellation of the word $\prod_{i=1}^{m} \overline{T}_{iR_{iT}} i$.

C_q. If $m \leqslant q$, $j < k < p$ (respectively, $j > k > p$), and R_j cancels with R_k and with R_p under some method of cancellation of the word $\prod_{i=1}^{m} \overline{T}_{iR_{iT}} i$, then T_j (respectively, $\overline{T}_j$) also cancels with R_k.

D_q. If $m \leqslant q$, then under any method of cancellation of the word $\prod_{i=1}^{m} \overline{T}_{iR_{iT}} i$ an adjacent n-tuple remains.

Lemma 1. $(A_q \;\&\; B_q \;\&\; C_q) \to B_{q+1}$.

Lemma 2. $(A_q \;\&\; B_{q+1}) \to A_{q+1}$.

Lemma 3. $D_q \to C_{q+1}$.

Lemma 4. $(A_{q+1} \;\&\; B_{q+1} \;\&\; C_{q+1}) \to D_{q+1}$.

For the proof of Lemmas 1 and 4 the following lemma is applied, valid for any product $\prod_{i=1}^{m} \overline{T}_{iR_{iT}} i$ in any group satisfying property 4):

Lemma 5. *If $i < j < k$, then it is impossible that T_j cancels with R_k, $\overline{T}_j$ cancels with R_i, and R_j cancels both with R_i and with R_k.*

Now, on the basis of mathematical induction, A_q, B_q, C_q, and D_q are true for all natural numbers q. We easily get rid of the assumption that $\leqslant 2R_i$ cancel with each R_j on each side, and Theorem 2 follows from Theorem 1 of paper ([2]).

§ 3. For the proof of Theorem 1 it is necessary to define the following sets:

$$M_1 = \{\text{all } R_i \text{ and all } R'_j \text{ such that } R_j \cong R'_j R''_j,\ R_k \cong \overline{R''_j} R'_k,\ \overline{R_j} \not\equiv R_k\}.$$

Suppose that M_n has already been defined.

$$\begin{aligned} M_{n+1} = \{&\text{all } R'_{i_1} \dots R'_{i_n} R'_{i_{n+1}} \text{ such that } R_{i_j} \cong R'_{i_j} R''_{i_j}\ (1 \le j \le n-1), \\ &R_{i_n} \cong R'''_{i_n} R'_{i_n} R''_{i_n}, \quad R_{i_{n+1}} \cong \overline{R''_{i_n}} R'_{i_{n+1}} R''_{i_{n+1}}, \\ &R'_{i_1} \dots R'_{i_{n-1}} R'_{i_n} R'''_{i_n} \in M_n, \quad \overline{R''_{i_n}} R'_{i_{n+1}} \in M_1, \quad \overline{R_{i_n}} \not\equiv R_{i_{n+1}}\}. \end{aligned}$$

$$M = \bigcup_{i=1}^{\infty} M_i.$$

Lemma 6. If a cyclically irreducible nonempty word $W = 1$ in the group G, and W is written on a circumference, then the resulting circular word C contains two nonintersecting words from the set M.

The fact that C contains two words from M follows directly from Theorem 2, since every adjacent n-tuple belongs to the set M_n. The proof of nonintersection presents certain technical difficulties.

Theorem 1 is proved with the help of Lemma 6 and the definition of the group G.

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REFERENCES

[1] M. Dehn, Math. Ann., **72**, 413 (1912). [2] M. Greendlinger, Comm. Pure and Appl. Math., **13**, 641 (1960).

Note: Figure translations are in progress. See original paper for figures.